UNIT-V

BOUNDARY VALUE PROBLEMS IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

① Solve by finite difference method, the boundary value problem $y''(x) - y(x) = 2$ where $y(0) = 0$ and $y(1) = 1$, taking $h = 1/4$

Soln

Given: $y''(x) - y(x) = 2$ —— ①

using the Central difference approximation, we have

$$y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

$$\therefore (1) \Rightarrow \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - y_i = 2$$

$$y_{i-1} - [2 + h^2]\, y_i + y_{i+1} = 2h^2$$

$$y_{i-1} - [2 + 1/16]\, y_i + y_{i+1} = 2\left[\frac{1}{16}\right]$$

$$y_{i-1} - \frac{33}{16} y_i + y_{i+1} = 1/8$$

$$y_{i-1} - 2.0625 y_i + y_{i+1} = 0.125 \quad \text{—— ②}$$

The boundary conditions are $y_0 = y(0) = 0$, $y_4 = y(1) = 1$

(ii) To find $y_1 = y(1/4)$, $y_2 = y(2/4)$, $y_3 = y(3/4)$

$i = 1$ in (2) $\Rightarrow y_0 - 2.0625 y_1 + y_2 = 0.125$

$$-2.0625 y_1 + y_2 = 0.125 \quad [\because y_0 = 0] \quad \text{—— ③}$$

$i = 2$ in (2) $\Rightarrow y_1 - 2.0625 y_2 + y_3 = 0.125$ —— (4)

$i = 3$ in (2) $\Rightarrow y_2 - 2.0625 y_3 + y_4 = 0.125$

$$y_2 - 2.0625 y_3 + 1 = 0.125$$

$$y_2 - 2.0625 y_3 = -0.875 \quad \text{—— ⑤}$$

(3) $\times 1 \Rightarrow -2.0625 y_1 + y_2 = 0.125$

(4) $\times (2.0625) \Rightarrow 2.0625 y_1 - 4.2539 y_2 + 2.0625 y_3 = 0.2578$

(+) $-3.2539 y_2 + 2.0625 y_3 =$ [illegible] —— ⑥

$$(5) \times 3.2539 \Rightarrow \quad 3.2539 y_2 - 6.7112 y_3 = -2.8472$$

$$(6) \times 1 \Rightarrow \quad -3.2539 y_2 + 2.0625 y_3 = 0.3828$$

$$(+) \quad -4.6487 y_3 = -2.4644$$

$$y_3 = \frac{2.4644}{4.6487} = 0.5301$$

$$(5) \Rightarrow y_2 = 2.0625 y_3 - 0.875$$

$$= (2.0625)(0.5301) - 0.875 = 0.2183$$

$$(4) \Rightarrow y_1 = 2.0625 y_2 - y_3 + 0.125$$

$$= (2.0625)(0.2183) - (0.5301) + 0.125 = 0.0451$$

Hence the result is

$y_0 = y(0) = 0$ [Given]

$y_1 = y(1/4) = 0.0451$

$y_2 = y(2/4) = 0.2183$

$y_3 = y(3/4) = 0.5301$

$y_4 = y(4/4) = y(1) = 1$ [Given]

(2) Solve the equation

$y''(x) - xy(x) = 0$ for $y(x_i), x_i = 0, 1/3, 2/3$, given

$y(0) + y'(0) = 1$ and $y(1) = 1$

Soln: The finite difference equation is

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = x_i y_i$$

(ie) $y_{i-1} - (2 + \frac{1}{9} x_i) y_i + y_{i+1} = 0$ ——(1)

Since $h = 1/3$

putting $i = 0, 1, 2$ in (1) we have

$y_{-1} - 2y_0 + y_1 = 0$ ——(2)

$y_0 - \frac{55}{27} y_1 + y_2 = 0$ ——(3)

and $y_1 - \frac{56}{27} y_2 + y_3 = 0$ —— ④

The first boundary condition is

$$y_0 + \frac{y_1 - y_{-1}}{2h} = 1 \qquad \left(\because y_1' = \frac{y_{i+1} - y_{i-1}}{2h} \right)$$

$$2y_0 + 3(y_1 - y_{-1}) = 2$$

$$y_{-1} = \frac{2y_0 + 3y_1 - 2}{3} \quad \text{—— ⑤}$$

The second boundary condition is

$$y_3 = 1$$

using (5) and (6) in equations (2), (3) and (4) we have

$$-2y_0 + 3y_1 = 1 \quad \text{—— ⑦}$$

$$y_0 - \frac{57}{27} y_1 + y_2 = 0 \quad \text{—— ⑧}$$

and $y_1 - \frac{56}{27} y_2 + 1 = 0$ —— ⑨

solving equations (7), (8) and (9) we get

$$y_0 = y(0) = -\frac{82}{83} = -0.9880$$

$$y_1 = y(1/3) = \frac{-27}{83} = -0.3253$$

$$y_2 = y(2/3) = \frac{27}{83} = 0.3253$$

Finite Difference Solution of one dimensional Heat equation By implicit and Explicit methods

classification of partial differential equation of Second order

	Elliptic Type	parabolic Type	Hyperbolic Type
1.	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Laplace Eqn in two dimension)	$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}$ (one dimensional heat equation)	$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial^2 u}{\partial t^2}$ (one dimensional wave equation)
2.	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ (poisson's equation)		

Example:- Classify the following partial equations:-

(i) $u_{xx} + 4u_{xy} + (x^2+4y^2)u_{yy} = \sin(x+y)$

(ii) $(x+1)u_{xx} - 2(x+2)u_{xy} + (x+3)u_{yy} = 0$

Soln:-

(i) Here, $A = 1$, $B = 4$, $C = (x^2+4y^2)$

$$B^2 - 4AC = 16 - 4(x^2+4y^2)$$

$$= 4[4 - x^2 - 4y^2]$$

The equation is elliptic if $4 - x^2 - 4y^2 < 0$

$$x^2 + 4y^2 > 4$$

$$\frac{x^2}{4} + \frac{y^2}{1} > 1$$

$\therefore$ It is elliptic in the region outside the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

It is hyperbolic inside, the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

It is parabolic on the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

(ii) Here, $A = x+1$, $B = -2(x+2)$, $C = x+3$

$$B^2 - 4AC = 4(x+2)^2 - 4(x+1)(x+3)$$

$$= 4[1] = 4 > 0$$

$\therefore$ The equation is hyperbolic at all points of the region.

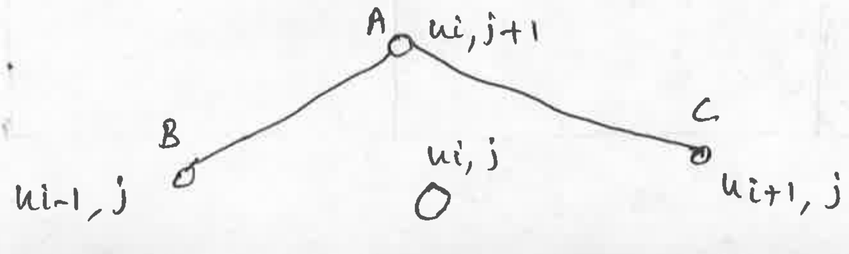
BENDER - SCHMIDT'S DIFFERENCE EQUATION

Corresponding to the parabolic equation $\left(\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}\right)$ (one dimensional heat equation) [Explicit method]

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1\,j} + u_{i+1\,j}\right]$$

when $\lambda = \frac{1}{2} = \frac{k}{ah^2}$ (ie) $k = \frac{a}{2}h^2$

This is valid only if $k = \frac{a}{2}h^2$



① Solve $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t} = 0$ given $u(0,t) = 0$, $u(4,t) = 0$, $u(x,0) = x(4-x)$.
Assume $h=1$. Find the values of u upto $t = 5$.

Soln:
$u_{xx} = a u_t \quad \therefore a = 2$

To use Bender-Schmidt's equation, $k = \frac{a}{2} h^2 = 1$

Step-size in time $= k = 1$. The values of $u_{i,j}$ are tabulated below.

x - direction →

$j \backslash i$	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0

↓ t-direction

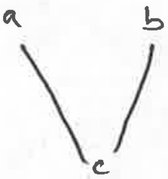
← $u(x,0) = x(4-x)$

The range for x is $(0, 4)$; for $t: (0, 5)$

$u(x,0) = x(4-x)$ This gives $u(0,0) = 0$, $u(1,0) = 3$, $u(2,0) = 4$, $u(3,0) = 3$, $u(4,0) = 0$.

For all t, $x=0$, $u=0$ and for all t at $x = 4$, $u=0$

using these values we fill up column under $x=0$, $x=4$ and row against $t=0$



This means $c = \frac{a+b}{2}$

The values of u at $t=1$ are written by seeing the values of u at $t=0$ and using the average formula.

②. Given $\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} = 0$, $f(0,t) = f(5,t) = 0$, $f(x,0) = x^2(25-x^2)$, find f in the range taking $h = 1$ and upto 5 seconds.

Soln:
By using method, $k = \frac{a}{2} h^2$

Here $a=1$, $h=1$, $\therefore k = \frac{1}{2}$

Step-size of time $= 1/2$

Step-size of $x = 1$.

$f(0,0) = 0, \quad f(1,0) = 24, \quad f(2,0) = 84, \quad f(3,0) = 144, \quad f(4,0) = 144, \quad f(5,0) = 0$

we have

$$u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right)$$

→ x direction

↓ t-direction

j \ i	0	1	2	3	4	5
0	0	24	84	144	144	0
1/2	0	42	84	114	72	0
1	0	42	78	78	57	0
1.5	0	39	60	67.5	39	0
2	0	30	53.25	49.5	33.75	0
2.5	0	26.625	39.75	43.5	24.75	0
3	0	19.875	35.0625	32.25	21.75	0
3.5	0	17.5312	26.0625	28.4062	16.125	0
4	0	13.0312	22.9687	21.0938	14.2031	0
4.5	0	11.4843	17.0625	18.5859	10.5469	0
5	0	8.5312	15.0351	13.8047	9.2929	0

CRANK-NICOLSON'S DIFFERENCE EQUATION, CORRESPONDING TO THE PARABOLIC EQUATION.

Formula

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right]$$

① Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ in $0 < x < 5$ $t \geq 0$ given that $u(x,0) = 20$, $u(0,t) = 0$ $u(5,t) = 100$. Compute u for the time-step with $h = 1$ by Crank-Nicholson method.

Soln:-

Here $a = 1$ $h = 1$ $\lambda = \dfrac{k}{ah^2}$

A convenient choice of λ makes the Crank-Nicholson difference scheme simple. Setting $\lambda = 1$, (ie) $k = ah^2$, $k = 1$

The Crank - Nicholson formula reduces to

$$u_{i,\,j+1} = \frac{1}{4}\left[u_{i-1,\,j+1} + u_{i+1,\,j+1} + u_{i-1,\,j} + u_{i+1,\,j}\right]$$

j \ i	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100

$$u_1 = \frac{1}{4}\left[0 + 20 + 0 + u_2\right]$$

$$= \frac{1}{4}\left[20 + u_2\right]$$

$$4u_1 = 20 + u_2$$

$$4u_1 - u_2 = 20 \quad \text{—— (1)}$$

$$u_2 = \frac{1}{4}\left[20 + 20 + u_1 + u_3\right]$$

$$= \frac{1}{4}\left[40 + u_1 + u_3\right]$$

$$4u_2 = 40 + u_i + u_3$$

$$u_1 + u_3 - 4u_2 = 40 \quad \text{—— (2)}$$

$$u_1 - 4u_2 + u_3 = -40 \quad \text{—— (2)}$$

$$u_3 = \frac{1}{4}\left[20 + 20 + u_2 + u_4\right]$$

$$4u_3 = 40 + u_2 + u_4$$

$$u_2 - 4u_3 + u_4 = -40 \quad \text{—— (3)}$$

$$u_4 = \frac{1}{4}\left[20 + 100 + u_3 + 100\right]$$

$$4u_4 = 220 + u_3$$

$$u_3 - 4u_4 = -220 \quad \text{—— (4)}$$

(1) × 1 ⇒ $4u_1 - u_2 = 20$

(2) × 4 ⇒ $4u_1 - 16u_2 + 4u_3 = -160$

(−) $15u_2 - 4u_3 = 180$ —— (5)

(3) × 4 ⇒ $4u_2 - 16u_3 + 4u_4 = -160$

(4) × 1 ⇒ $u_3 - 4u_4 = -220$

(+) $4u_2 - 15u_3 = -380$ —— (6)

⑤ $\times 4 \Rightarrow$ $60u_2 - 16u_3 = 720.$

⑧ $\times 15 \Rightarrow$ $60u_2 - 225u_3 = -5700$

(−) $209u_3 = 6420$

$$u_3 = 30.72$$

(4) $\Rightarrow$ $4u_4 = 220 + u_3 = 220 + 30.72$

$$u_4 = 62.68$$

(3) $\Rightarrow$ $u_2 = 4u_3 - u_4 - 40$

$= 4(30.720) - 62.68 - 40$

$= 20.2$

(1) $\Rightarrow$ $u_1 = \frac{1}{4}[20 + u_2]$

$= \frac{1}{4}[20 + 20.2]$

$= 10.05$

$\therefore$ The values are 10.05, 20.2, 30.72, 62.68

② Solve by Crank-Nicholson method the equation $u_{xx} = u_t$ subject to $u(x,0) = 0$, $u(0,t) = 0$ and $u(1,t) = t$, for two time steps.

Soln:

x ranges from 0 to 1 take $h = \frac{1}{4}$; here $a = 1$.

$\therefore k = ah^2$ to use simple form

$$k = 1\left(\tfrac{1}{4}\right)^2 = \frac{1}{16}$$

we use $u_{i,j+1} = \frac{1}{4}[u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j} + u_{i+1,j}]$ —①

$x \rightarrow$ direction

$t \downarrow$

	0	0.25	0.5	0.75	1
0	0	0	0	0	0
$\frac{1}{16}$	0	u_1	u_2	u_3	$\frac{1}{16}$
$\frac{2}{16}$	0	u_4	u_5	u_6	$\frac{2}{16}$
$\frac{3}{16}$	0				$\frac{3}{16}$

Let the unknowns be represented by $u_1, u_2, u_3 \ldots$

The boundary conditions are marked in the table against $t=0$, $x=0$, and $x=1$

using the scheme (1),

$u_1 = \frac{1}{4}(0+0+0+u_2)$ $\qquad u_1 = \frac{1}{4}u_2$

$u_2 = \frac{1}{4}(0+0+u_1+u_3)$ $\qquad u_2 = \frac{1}{4}(u_1+u_3)$

$u_3 = \frac{1}{4}(0+0+u_2+\frac{1}{16})$ $\qquad u_3 = \frac{1}{4}(u_2+\frac{1}{16})$

Solving the three equations given by (2), (3) (4) we get u_1, u_2, u_3

Substitute u_3, u_1 values in (3)

$$u_2 = \frac{1}{4}\left[\frac{1}{4}u_2 + \frac{1}{4}\left(u_2+\frac{1}{16}\right)\right]$$

$$u_2 = \frac{1}{224}\ 0.0045, \quad u_1 = \frac{1}{896} = 0.0011, \quad u_3 = 0.0168$$

Similarly u_4, u_5, u_6 can be got again getting 3 equations in 3 unknowns u_4, u_5, u_6.

we get $u_4 = 0.005899$ $\quad u_5 = 0.01913$, $\quad u_6 = 0.05277$

In solving the three equations (2), (3), (4) we could have used Gauss-Seidel method also. The iterated values are noted below.

u_1	-	0.125	0.0391	0.0059	0.0017	0.0012	0.0011	0.0011
u_2	0.5	0.1563	0.0235	0.0069	0.0048	0.0045	0.0045	0.0045
u_3	0.5	0.0547	0.0215	0.0174	0.0168	0.0168	0.0168	0.0168

ONE DIMENSIONAL WAVE EQUATION.

Formula :- $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

① Solve $y_{tt} = y_{xx}$ upto $t = 0.5$ with a spacing of 0.1 subject to $y(0,t) = 0$ $y(1,t) = 0$, $y_t(x,0) = 0$ and $y(x,0) = 10 + x(1-x)$.

Soln:

Here $a = 1$, $h = 0.1$, $k = \frac{h}{a} = 0.1$

Formula, $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

From $y(0,t) = 0 \Rightarrow$ y along $x=0$ are all zero

From $y(1,t) = 0 \Rightarrow$ y along $x=1$ are all zero.

$y(x,0) = 10 + x(1-x)$

$y(0.1,0) = 10 + (0.1)(1-0.1) = 10.09$

$y(0.2,0) = 10 + (0.2)(1-0.2) = 10.16$

$y(0.3,0) = 10 + (0.3)(1-0.3) = 10.21$

$y(0.4,0) = 10 + (0.4)(1-0.4) = 10.24$

$y(0.5,0) = 10 + (0.5)(1-0.5) = 10.25$

$y(0.6,0) = 10 + (0.6)(1-0.6) = 10.24$

$y(0.7,0) = 10 + (0.7)(1-0.7) = 10.21$

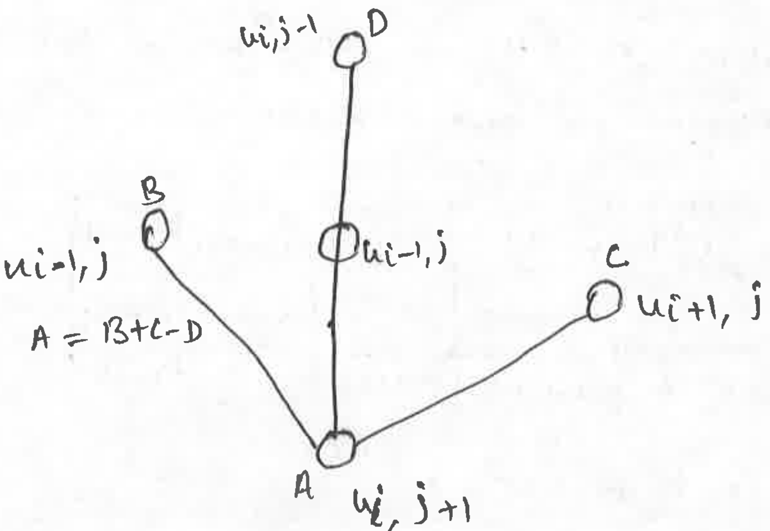
$y(0.8,0) = 10 + (0.8)(1-0.8) = 10.16$

$y(0.9,0) = 10 + (0.9)(1-0.9) = 10.09$

Now, we fill up the row $t=0$ using the above values

$$u_t(x,0) = 0 \Rightarrow u_{i,1} = \frac{u_{i+1,0} + u_{i-1,0}}{2}$$

Now we draw the table for what we require



$\rightarrow x$

$\downarrow t$

$t \backslash x$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0	10.09	10.16	10.21	10.24	10.25	10.24	10.21	10.16	10.09	10.0
0.1	0	5.08	10.15	10.20	10.23	10.24	10.23	10.20	10.15	10.08	0.0
0.2	0	0.06	5.12	10.17	10.20	10.21	10.20	10.17	10.12	0.06	0.0
0.3	0	0.04	0.08	5.12	10.15	10.16	10.15	10.12	10.08	0.04	0.0
0.4	0	0.02	0.04	0.06	5.08	10.09	10.08	10.06	10.04	0.02	0.0
0.5	0	0	0	0	0	0	0	0	0	0	0

5. Approximate the solution to the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $0 < x < 1$, $t > 0$, $u(0,t) = u(1,t) = 0$, $t > 0$, $u(x,0) = \sin 2\pi x$, $0 \le x \le 1$ and $\frac{\partial u}{\partial t}(x,0) = 0$, $0 \le x \le 1$ with $\Delta x = 0.25$ and $\Delta t = 0.25$ for 3 time steps.

Soln:

Here $a=1$, $h=0.25$, $k=0.25$

0.25 \ 0.25	0	0.25	0.5	0.75	1
0	0	1	0	-1	0
0.25	0	0	0	0	0
0.5	0	-1	0	1	0
0.75	0	0	0	0	0
1	0	1	0	-1	0

$u(x,0) = \sin 2\pi x$

$u(0.25, 0) = \sin \pi/2 = 1$

$u(0.5, 0) = \sin \pi = 0$

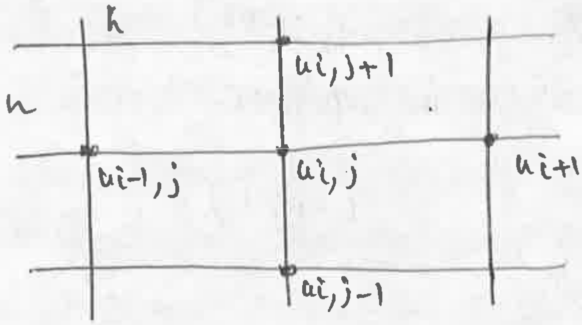
$u(0.75, 0) = \sin \frac{3\pi}{2} = -1$

TWO DIMENSIONAL LAPLACE EQUATION

ELLIPTIC EQUATIONS

$\therefore u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$

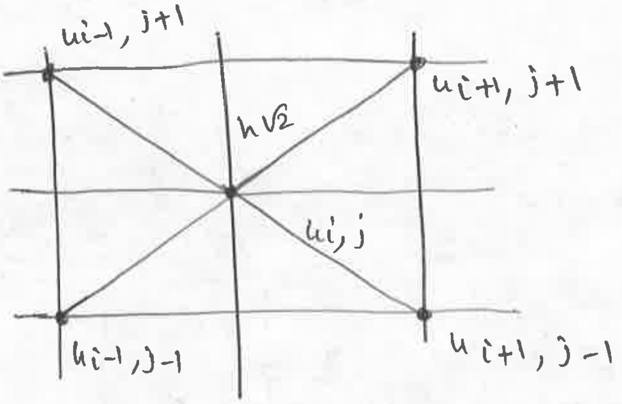
This is called standard five point formula



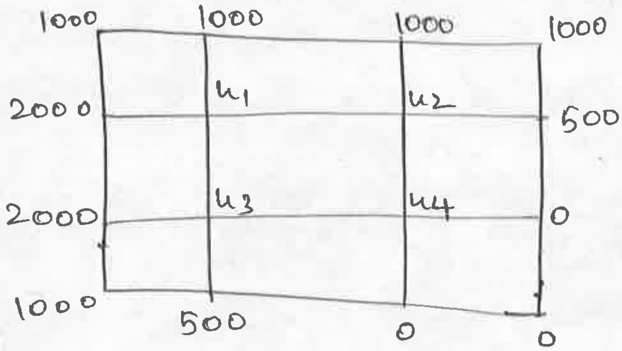
Central value = Average of the other four values.

Diagonal five-point formula

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}\right]$$



① Obtain a finite difference scheme to solve the Laplace equation. Solve $\nabla^2 u = 0$ at the pivotal points in the square shown fitted with square mesh. Use Leibmann's iteration procedure. (5 iteration only)



Solution:-

We can assume some value for u_4 (or any other u) and proceed iterative procedure; we can take $u_4 = 0$ and proceed (or) we take a value of $u_4 = 400$ (Guess this seeing the values of u on the vertical line through u_2, u_4).

Rough values:-

$u_1 = (1000 + 2000 + 1000 + 400)/4 = 1100$ (DFPF)

$u_2 = \frac{1}{4}(u_1 + u_4 + 1500) = 750$ (SFPF)

$u_3 = \frac{1}{4}(u_1 + u_4 + 2500) = 1000$ (SFPF)

$u_4 = \frac{1}{4}(u_2 + u_3) = 437.5$ (SFPF)

First iteration: Here after we apply only SFPF

$$u_1^{(1)} = \frac{1}{4}(750 + 1000 + 3000) = 1187.5$$

$$u_2^{(1)} = \frac{1}{4}(1187.5 + 437.5 + 1500) = 781.25$$

$$u_3^{(1)} = \frac{1}{4}(1187.5 + 437.5 + 2500) = 1031.25$$

$$u_4^{(1)} = \frac{1}{4}(781.25 + 1031.25) = 453.125$$

Second iteration:-

$$u_1^{(2)} = \frac{1}{4}(781.25 + 1031.25 + 3000) = 1203.125$$

$$u_2^{(2)} = \frac{1}{4}(1203.125 + 453.125 + 1500) = 789.1$$

$$u_3^{(2)} = \frac{1}{4}(1203.125 + 453.125 + 2500) = 1039.1$$

$$u_4^{(2)} = \frac{1}{4}(789.1 + 1039.1) = 457.1$$

Third iteration:-

$$u_1^{(3)} = \frac{1}{4}(789.1 + 1039.1 + 3000) = 1207.1$$

$$u_2^{(3)} = \frac{1}{4}(1207.1 + 457.1 + 1500) = 791.1$$

$$u_3^{(3)} = \frac{1}{4}(1207.1 + 457.1 + 2500) = 1041.1$$

$$u_4^{(3)} = \frac{1}{4}(791.1 + 1041.1) = 458.1$$

Fourth iteration:-

$$u_1^{(4)} = \frac{1}{4}(791.1 + 1041.1 + 3000) = 1208.1$$

$$u_2^{(4)} = \frac{1}{4}(1208.1 + 458.1 + 1500) = 791.6$$

$$u_3^{(4)} = \frac{1}{4}(1208.1 + 458.1 + 2500) = 1041.6$$

$$u_4^{(4)} = \frac{1}{4}(791.6 + 1041.6) = 458.3$$

Fifth iteration:-

$$u_1^{(5)} = \frac{1}{4}(791.6 + 1041.6 + 3000) = 1208.3$$

$$u_2^{(5)} = \frac{1}{4}(1208.3 + 458.3 + 1500) = 791.7$$

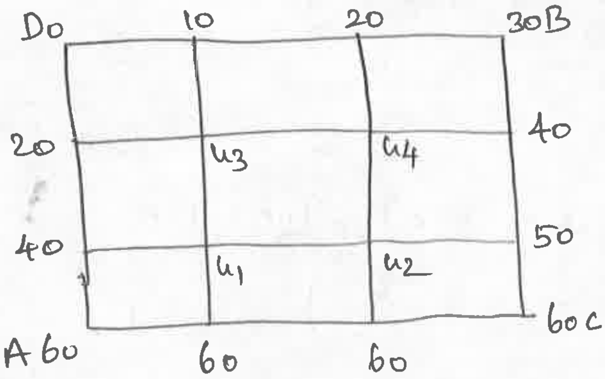
$$u_3^{(5)} = \frac{1}{4}(1208.3 + 458.3 + 2500) = 1041.7$$

$$u_4^{(5)} = \frac{1}{4}(791.7 + 1041.7) = 458.7$$

we are getting result correct to one decimal place. Further the increases in the value is < 0.1

$\therefore u_1 = 1208.1, \ u_2 = 791.7, \ u_3 = 1041.7, \ u_4 = 458.4$

⑤ Solve:- $\Delta^2 u = 0$, the boundary conditions are given below. (give only three iteration).



Rough values: Let $u_4 = 0$

$u_1 = \frac{1}{4}[60 + u_4 + 20 + 60] = 35$ (DFPF)

$u_2 = \frac{1}{4}[u_1 + u_4 + 50 + 60] = 36.25$ (SFPF)

$u_3 = \frac{1}{4}[10 + 20 + u_1 + u_4] = 16.25$ (SFPF)

$u_4 = \frac{1}{4}[20 + u_3 + u_2 + 40] = 28.125$ (SFPF)

First iteration [Here after we apply only SFPF]

$u_1^{(1)} = \frac{1}{4}[40 + 60 + 16.25 + 36.25] = 38.125$

$u_2^{(1)} = \frac{1}{4}[60 + 50 + 38.125 + 28.125] = 44.0625$

$u_3^{(1)} = \frac{1}{4}[20 + 10 + 38.125 + 28.125] = 24.0625$

$u_4^{(1)} = \frac{1}{4}[20 + 40 + 24.0625 + 44.0625] = 32.0313$

Second iteration:-

$u_1^{(2)} = \frac{1}{4}[60 + 40 + 24.0625 + 44.0625] = 42.0313$

$u_2^{(2)} = \frac{1}{4}[60 + 50 + 42.0313 + 32.0313] = 46.0157$

$u_3^{(2)} = \frac{1}{4}[20 + 10 + 42.0313 + 32.0313] = 26.0157$

$u_4^{(2)} = \frac{1}{4}[20 + 40 + 26.0157 + 46.0157] = 33.0079$

Third iteration:-

$$u_1^{(3)} = \frac{1}{4}[60+40+46.0157+26.0157] = 43.0079$$

$$u_2^{(3)} = \frac{1}{4}[60+50+43.0079+33.0079] = 46.5040$$

$$u_3^{(3)} = \frac{1}{4}[20+10+43.0079+33.0079] = 26.5040$$

$$u_4^{(3)} = \frac{1}{4}[20+40+46.5040+26.5040] = 33.252$$

(3) solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in $|x| < 1$, $|y| < 1$ with $h = \frac{1}{2}$ and

$u(x \pm 1) = x^2$, $u(\pm 1, y) = y^2$

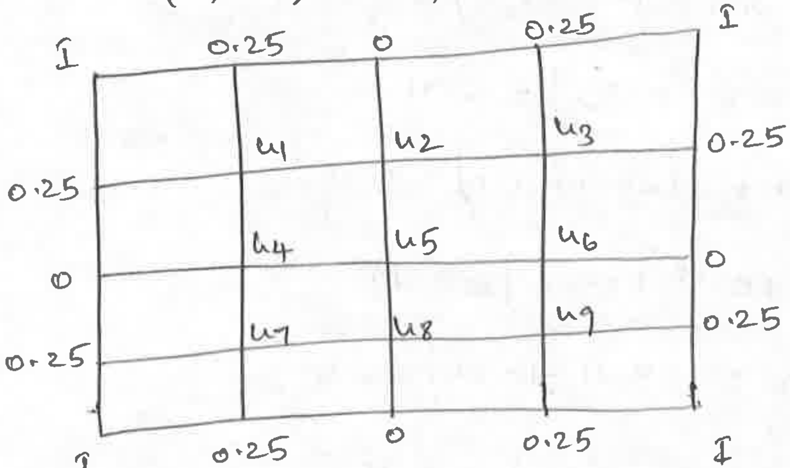
(i) $u(x, 1) = x^2$ $\quad -1 < x < 1$

(ii) $u(x, -1) = x^2$, $\quad -1 < x < 1$

(iii) $u(1, y) = y^2$, $\quad -1 < y < 1$

(iv) $u(-1, y) = y^2$, $\quad -1 < y < 1$

Soln:- Given $u(x, \pm 1) = x^2$, $u(\pm 1, y) = y^2$



Let the values of u at the 9 interior grid points be u_1, u_2, u_3 .... u_9

We use Liebman's iteration method to find the value of $u_1, u_2, u_3 \ldots u_9$

Step 1:- To find the 9 rough values of $u_1, u_2, u_3 \ldots u_9$.

$$u_5 = \frac{1}{4}[0+0+0+0] = 0 \; \{SFPF\}$$

$$u_1 = \frac{1}{4}[0+0+0+1] = \frac{1}{4} \; [DFPF]$$

$$u_3 = \frac{1}{4}[1+0+0+0] = \frac{1}{4}[DFPF]$$

$$u_7 = \frac{1}{4}[1+0+0+0] = \frac{1}{4} \quad [DFPF]$$

$$u_9 = \frac{1}{4}[1+0+0+0] = \frac{1}{4} \quad [DFPF]$$

Now we find the other four values using [SFPF]

$$u_2 = \frac{1}{4}[0+0+0.25+0.25] = 0.125$$

$$u_4 = \frac{1}{4}[0+0+0.25+0.25] = 0.125$$

$$u_6 = \frac{1}{4}[0+0+0.25+0.25] = 0.125$$

$$u_8 = \frac{1}{4}[0+0+0.25+0.25] = 0.125$$

Step:2 First iteration :- In all further calculations we use SFPF and the latest available values

$$u_1^{(1)} = \frac{1}{4}[0.25+0.25+0.125+0.125] = 0.19$$

$$u_2^{(1)} = \frac{1}{4}[0+0.19+0.25+0] = 0.11$$

$$u_3^{(1)} = \frac{1}{4}[0.25+0.25+0.11+0.125] = 0.18$$

$$u_4^{(1)} = \frac{1}{4}[0+0.19+0+0.25] = 0.11$$

$$u_5^{(1)} = \frac{1}{4}[0.11+0.11+0.125+0.125] = 0.12$$

$$u_6^{(1)} = \frac{1}{4}[0+0.12+0.18+0.25] = 0.14$$

$$u_7^{(1)} = \frac{1}{4}[0.25+0.25+0.11+0.25] = 0.18$$

$$u_8^{(1)} = \frac{1}{4}[0+0.12+0.18+0.25] = 0.14$$

$$u_9^{(1)} = \frac{1}{4}[0.25+0.25+0.14+0.14] = 0.20$$

Second iteration:-

$$u_1^{(2)} = \frac{1}{4}[0.25+0.25+0.11+0.11] = 0.18$$

$$u_2^{(2)} = \frac{1}{4}[0+0.18+0.12+0.18] = 0.12$$

$$u_3^{(2)} = \frac{1}{4}[0.25+0.25+0.12+0.14] = 0.19$$

$$u_4^{(2)} = \frac{1}{4}[0+0.18+0.12+0.18] = 0.12$$

$$u_5^{(2)} = \frac{1}{4}[0.12+0.12+0.14+0.14] = 0.13$$

$$u_6^{(2)} = \frac{1}{4}[0.13+0+0.19+0.2] = 0.13$$

$$u_7^{(2)} = \frac{1}{4}[0.25+0.12+0.14+0.25] = 0.19$$

$$u_8^{(2)} = \frac{1}{4}[0+0.19+0.2+0.13] = 0.13$$

$$u_9^{(2)} = \frac{1}{4}[0.25+0.25+0.13+0.13] = 0.19$$

Third iteration:-

$$u_1^{(3)} = \frac{1}{4}[0.25+0.25+0.12+0.12] = 0.19$$

$$u_2^{(3)} = \frac{1}{4}[0+0.19+0.13+0.19] = 0.13$$

$$u_3^{(3)} = \frac{1}{4}[0.25+0.25+0.13+0.13] = 0.19$$

$$u_4^{(3)} = \frac{1}{4}[0+0.19+0.13+0.19] = 0.13$$

$$u_5^{(3)} = \frac{1}{4}[0.13+0.13+0.13+0.13] = 0.13$$

$$u_6^{(3)} = \frac{1}{4}[0.13+0+0.19+0.19] = 0.13$$

$$u_7^{(3)} = \frac{1}{4}[0.25+0.13+0.13+0.25] = 0.19$$

$$u_8^{(3)} = \frac{1}{4}[0+0.19+0.19+0.13] = 0.13$$

$$u_9^{(3)} = \frac{1}{4}[0.25+0.25+0.13+0.13] = 0.19.$$

Fourth iteration:-

$$u_1^{(4)} = \frac{1}{4}[0.25+0.25+0.13+0.13] = 0.19$$

$$u_2^{(4)} = \frac{1}{4}[0+0.19+0.13+0.19] = 0.13$$

$$u_3^{(4)} = \frac{1}{4}[0.25+0.25+0.13+0.13] = 0.19$$

$$u_4^{(4)} = \frac{1}{4}[0+0.19+0.13+0.19] = 0.13$$

$$u_5^{(4)} = \frac{1}{4}[0.13+0.13+0.13+0.13] = 0.13$$

$$u_6^{(4)} = \frac{1}{4}[0.13+0+0.19+0.19] = 0.13$$

$$u_7^{(4)} = \frac{1}{4}[0.25+0.13+0.13+0.25] = 0.19$$

$$u_8^{(4)} = \frac{1}{4}[0 + 0.19 + 0.19 + 0.13] = 0.13$$

$$u_9^{(4)} = \frac{1}{4}[0.25 + 0.25 + 0.13 + 0.13] = 0.19$$

Step:- 3

Since in the third and fouth iterations all the values of $u_{i,j}$ at the grid points are same, the iteration process is stoped.

Hence

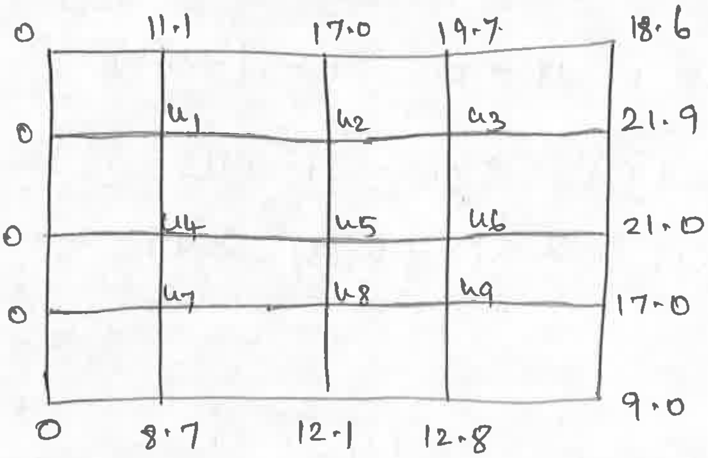
$u_1 = 0.19$ $u_2 = 0.13$, $u_3 = 0.19$

$u_4 = 0.13$, $u_5 = 0.13$ $u_6 = 0.13$

$u_7 = 0.19$ $u_8 = 0.13$ $u_9 = 0.19$

[correct to 2 places of decimals]

④ Find by the Liebmann's method the values at the interior lattice points of a square region of the harmonic function u whose boundary values are as shown in the following figure.



Solution:-

Since u is harmonic it satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in the square} \quad —①$$

Let the interior values of u at the grid points be $u_1, u_2, u_3 \ldots u_9$. we will find the values of u at the interior mesh points as explained in the previous article. we will first find the rough values of u and then proceed to refine them.

Finding rough values

$$u_5 = \frac{1}{4}[0+17.0+21.0+12.1] = 12.5 \quad (SFPF)$$

$$u_1 = \frac{1}{4}[0+12.5+0+17.0] = 7.4 \quad (DFPF)$$

$$u_3 = \frac{1}{4}[12.5+18.6+17.0+1.0] = 17.3 \quad (DFPF)$$

$$u_7 = \frac{1}{4}[12.5+0+0+12.1] = 6.2 \quad (DFPF)$$

$$u_9 = \frac{1}{4}[12.5+9.0+12.1+21.0] = 13.7 \quad (DFPF)$$

$$u_2 = \frac{1}{4}[17.0+12.5+7.4+17.3] = 13.6 \quad (SFPF)$$

$$u_4 = \frac{1}{4}[7.4+6.2+0+12.5] = 6.5 \quad (SFPF)$$

$$u_6 = \frac{1}{4}[12.5+21.0+17.3+13.7] = 16.1 \quad (SFPF)$$

$$u_8 = \frac{1}{4}[12.5+12.1+6.2+13.7] = 11.1 \quad (SFPF)$$

Now, we have got the rough values at all interior grid points and already we possess the boundary values at the lattice points. we will now improve the values by using always SFPF

First iteration: (we obtain all values by SFPF)

$$u_1^{(1)} = \frac{1}{4}[0+11.1+u_2+u_4] = \frac{1}{4}[0+11.1+13.6+6.5] = 7.8$$

$$u_2^{(1)} = \frac{1}{4}[17.0+12.5+7.8+17.3] = 13.7$$

$$u_3^{(1)} = \frac{1}{4}[13.7+21.9+19.7+16.1] = 17.9$$

$$u_4^{(1)} = \frac{1}{4}[0+12.5+7.8+6.2] = 6.6$$

$$u_5^{(1)} = \frac{1}{4}[13.7+11.1+6.6+16.1] = 11.9$$

$$u_6^{(1)} = \frac{1}{4}[17.9+13.7+11.9+21.0] = 16.1$$

$$u_7^{(1)} = \frac{1}{4}[6.6+8.7+0+11.1] = 6.6$$

$$u_8^{(1)} = \frac{1}{4}[11.9+12.1+6.6+13.7] = 11$$

$$u_9^{(1)} = \frac{1}{4}[16.1+12.8+17.0+11.1] = 14.3$$

Second iteration:-

$$u_1^{(2)} = \frac{1}{4}[0+11.1+13.7+6.6] = 7.9$$

$$u_2^{(2)} = \frac{1}{4}[17.0+17.9+7.9+11.9] = 13.7$$

$$u_3^{(2)} = \frac{1}{4}[13.7+21.9+19.7+16.1] = 17.9$$

$$u_4^{(2)} = \frac{1}{4}[0+11.9+7.9+6.6] = 6.6$$

$$u_5^{(2)} = \frac{1}{4}[13.7+11.1+6.6+16.1] = 11.9$$

$$u_6^{(2)} = \frac{1}{4}[17.9+14.3+11.9+21.0] = 16.3$$

$$u_7^{(2)} = \frac{1}{4}[6.6+8.7+0+11.1] = 6.6$$

$$u_8^{(2)} = \frac{1}{4}[11.9+12.1+6.6+14.3] = 11.2$$

$$u_9^{(2)} = \frac{1}{4}[16.3+12.8+17.0+11.2] = 14.3$$

Third iteration:-

$$u_1^{(3)} = \frac{1}{4}[0+11.1+13.7+6.6] = 7.9$$

$$u_2^{(3)} = \frac{1}{4}[17.0+17.9+7.9+11.9] = 13.7$$

$$u_3^{(3)} = \frac{1}{4}[13.7+21.9+19.7+16.3] = 17.9$$

$$u_4^{(3)} = 1/4\,[0+11.9+7.9+6.6] = 6.6$$

$$u_5^{(3)} = \frac{1}{4}[13.7+11.2+6.6+16.3] = 11.9$$

$$u_6^{(3)} = \frac{1}{4}[17.9+14.3+11.9+21.0] = 16.3$$

$$u_7^{(3)} = \frac{1}{4}[6.6+8.7+0+11.2] = 6.6$$

$$u_8^{(3)} = \frac{1}{4}[11.9+12.1+6.6+14.3] = 11.2$$

$$u_9^{(3)} = \frac{1}{4}[16.3+12.8+17.0+11.2] = 14.3$$

The third iteration values are same as the corresponding values of the second iteration. Hence we stop the procedure and accept.

$u_1 = 7.9$, $u_2 = 13.7$, $u_3 = 17.9$, $u_4 = 6.6$, $u_5 = 11.9$, $u_6 = 16.3$, $u_7 = 6.6$

$u_8 = 11.2$, $u_9 = 14.3$

TWO DIMENSIONAL POISSON EQUATION

Poisson equation

Any equation of the form $\nabla^2 u = f(x, y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \quad -(1)$$

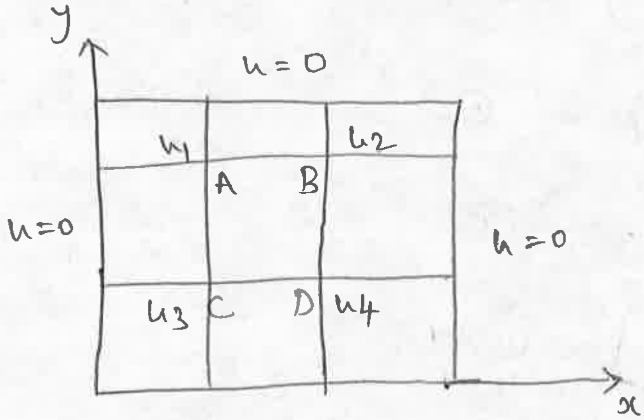
is called as poisson's equation where $f(x, y)$ is a function of x and y only

Formula

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh) \quad -(2)$$

(1) Solve the poisson equation $\nabla^2 u = -10(x^2+y^2+10)$ over the square with sides $x=0$, $y=0$, $x=3$ $y=3$ with $u=0$ on the boundary, taking $h=1$.

Soln:



Let the values of u at the four mesh points A, B C and D be u_1, u_2, u_3, u_4 respectively. The differential equation is

$$\nabla^2 u = -10(x^2+y^2+10) \quad -(1)$$

Replacing $\nabla^2 u$ by the finite difference expressions and putting $x = ih$, $y = ih$ $(h=1)$ in (1) we get

$$u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = -10(i^2+j^2+10) \quad -(2)$$

Applying the formula (1) at A [where $i=1$, $j=2$]

$$0 + 0 + u_2 + u_3 - 4u_1 = -10(1+4+10)$$

$$u_2 + u_3 - 4u_1 = -150 \quad -(3)$$

Applying the formula (1) at B where $i=2, j=2$

$$u_1+0+0+u_4-4u_2=-10(4+4+10)$$

$$u_1+u_4-4u_2=-180 \quad \text{④}$$

Applying the formula (1) at C where $i=1, j=1$

$$0+u_1+u_4+0-4u_3=-10(1+1+10)$$

$$u_1+u_4-4u_3=-120 \quad \text{⑤}$$

Applying the formula (1) at D where $i=2, j=1$

$$u_3+u_2+0+0-4u_4=-10(4+1+10)$$

$$u_3+u_2-4u_4=-150 \quad \text{⑥}$$

$$u_1=\frac{1}{4}\left[u_2+u_3+150\right] \quad \text{⑦}$$

$$u_2=\frac{1}{4}\left[u_1+u_4+180\right] \quad \text{⑧}$$

$$u_3=\frac{1}{4}\left[u_1+u_4+120\right] \quad \text{⑨}$$

$$u_4=\frac{1}{4}\left[u_2+u_3+150\right] \quad \text{⑩}$$

From (7) and (10)

we find that $u_4=u_1$

So it is enough if we find u_1, u_2, u_3

we start the iteration by putting $u_2=0=u_3$ in (7)

we get $u_1^{(1)}=\frac{150}{4}=37.5$

putting $u_1=37.5=u_4$ in (8) and (9) we get

$$u_2^{(1)}=\frac{1}{4}(75+180)=\frac{225}{4}=63.75$$

$$u_3^{(1)}=\frac{1}{4}(75+120)=\frac{195}{4}=48.7$$

For the second iteration, we have

$$u_1^{(2)} = \frac{1}{4}(63.75 + 48.75 + 150) = \frac{262.5}{4} = 65$$

$$u_2^{(2)} = \frac{1}{4}(65 + 65 + 180) = \frac{310}{4} = 80$$

$$u_3^{(2)} = \frac{1}{4}(65 + 65 + 120) = \frac{250}{4} = 60$$

For the third iteration, we have

$$u_1^{(3)} = \frac{1}{4}(80 + 60 + 150) = \frac{290}{4} = 70$$

$$u_2^{(3)} = \frac{1}{4}(70 + 70 + 180) = \frac{320}{4} = 80$$

$$u_3^{(3)} = \frac{1}{4}[70 + 70 + 120] = \frac{260}{4} = 65$$

For fourth iteration, we have

$$u_1^{(4)} = \frac{1}{4}(80 + 65 + 150) = \frac{295}{4} = 75$$

$$u_2^{(4)} = \frac{1}{4}(75 + 75 + 180) = \frac{330}{4} = 82.5$$

$$u_3^{(4)} = \frac{1}{4}(75 + 75 + 120) = \frac{270}{4} = 67.5$$

For the fifth iteration, we have

$$u_1^{(5)} = \frac{1}{4}(82 + 67.5 + 100) = \frac{300}{4} = 75$$

$$u_2^{(5)} = \frac{1}{4}(75 + 75 + 180) = \frac{330}{4} = 82.5$$

$$u_3^{(5)} = \frac{1}{4}(75 + 75 + 120) = \frac{270}{4} = 67.5$$

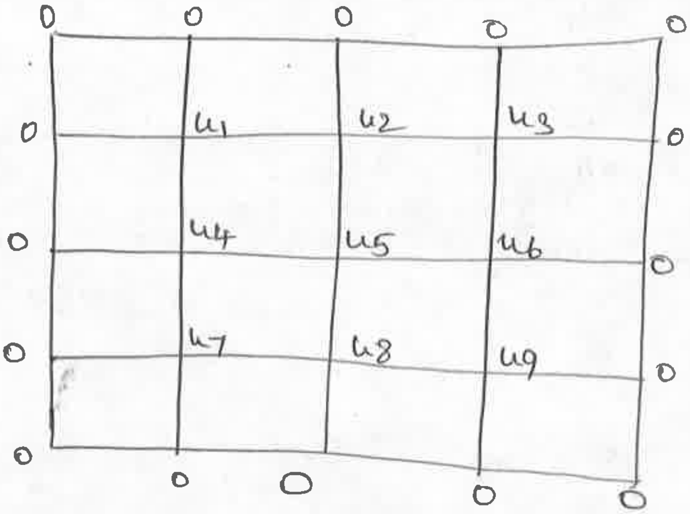
we find that these values are the same as in the fourth iteration.

$u_1 = 75$, $u_2 = 82.5$, $u_3 = 67.5$, $u_4 = 75$ [since $u_4 = u_1$]

5. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 8x^2y^2$ in the square mesh given $u=0$ on the four boundaries dividing the square into 16 subsquares of length 1 unit.

Solution:-

Here $h=1$. The region of solution of the given Laplace equation with the boundary values are given in the table.



Let $u_1, u_2, u_3 \ldots u_9$ be the values of u at the interior grid points.

Choose coordinate system with origin at the centre u_5 of the square mesh.

We note that the given poisson partial differential equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 8x^2y^2$ is symmetrical about x and y axes and also about the line $y=x$.

Hence we have $u_1 = u_3 = u_7 = u_9$ and $u_2 = u_4 = u_6 = u_8$

Hence we have to find u_1, u_2, u_5 only.

The standard five point formula for the given poisson equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} - 4u_{i,j} = 8i^2j^2 \quad \text{——} (1)$$

using (1) at u_7 $(i=-1, j=-1)$ we have

$$u_{-2,-1} + u_{0,-1} + u_{-1,-2} + u_{-1,0} - 4u_{-1,-1} = 8(-1)^2(-1)^2$$

(ie) $0 + u_8 + 0 + u_4 - 4u_7 = 8$

(ie) $u_2 + u_2 - 4u_1 = 8$

(ie) $u_2 - 2u_1 = 4 \quad \text{——} (2)$

using (1) at u_2 $(i=0, j=1)$ we have

$$u_{-1,1} + u_{1,1} + u_{0,0} + u_{0,2} - 4u_{0,1} = 8 \times 0 \times 1 = 0$$

(ie) $u_1 + u_3 + u_5 + 0 - 4u_2 = 0$

$\therefore 2u_1 - 4u_2 + u_5 = 0$ —— (3)

using (1) at u_5 $(i=0, j=0)$ we have

$$u_{-1,0} + u_{1,0} + u_{0,-1} + u_{0,1} - 4u_{0,0} = 0$$

$u_4 + u_6 + u_8 + u_2 - 4u_5 = 0$

$4u_2 - 4u_5 = 0$

$u_2 = u_5$ —— (4)

Solving these equations (2), (3), (4) we get

$u_1 = -3, \quad u_2 = -2, \quad u_5 = -2$

$\therefore$ The solution to the given poission equation at the 9 interior mesh points are

$u_1 = u_3 = u_7 = u_9 = -3$

$u_2 = u_4 = u_6 = u_8 = -2$ and $u_5 = -2$.