UNIT - IV

Initial value problems for ordinary differential equations

Taylor series method:

$$y_{n+1} = y_n + \frac{(x-x_0)}{1!} y_n' + \frac{(x-x_0)^2}{2!} y_n'' + \frac{(x-x_0)^3}{3!} y_n''' + \cdots$$

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \quad \text{where } h = (x - x_0)$$

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

Problems:

① Solve $y' = x + y$; $y(0) = 1$ by Taylor's series method. Find the value of y at $x = 0.1$ and $x = 0.2$

Soln:

$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$

Here, $x_0 = 0$, $y_0 = 1$

$y' = x + y$, $\quad y_0' = x_0 + y_0 = 0 + 1 = 1$

$y'' = 1 + y'$ $\quad y_0'' = 1 + y_0' = 1 + 1 = 2$

$y''' = y''$ $\quad y_0''' = y_0'' = 2$

$$\therefore y = y_0 + (x - x_0) y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \cdots$$

$$y = 1 + (x-0)(1) + \frac{(x-0)^2}{2!}(2) + \frac{(x-0)^3}{3!}(2) + \cdots$$

$$y = 1 + x + \frac{x^2}{2}(2) + \frac{x^3}{6}(2) + \cdots$$

$$y = 1 + x + x^2 + x^3/3 + \cdots$$

$$\therefore y(0.1) = 1 + (0.1) + (0.1)^2 + \frac{(0.1)^3}{3} = 1.1103$$

$$y(0.2) = 1 + (0.2) + (0.2)^2 + \frac{(0.2)^3}{3} = 1.2426$$

Another method

$x_0 = 0, \quad y_0 = 1, \quad h = 0.1$ since $h = x_1 - x_0 = 0.1 - 0 = 0$

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

$$y(0.1) = 1 + \frac{(0.1)}{1!}(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \cdots$$

$$= 1.1103$$

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

$y' = x + y$

$y_1' = x_1 + y_1 = 0.1 + 1.1103 = 1.2103$

$y_1'' = 1 + y_1' = 1 + 1.2103 = 2.2103$

$y_1''' = y_1'' = 2.2103$

$$\therefore y(0.2) = 1.1103 + \frac{(0.1)}{1!}(1.2103) + \frac{(0.1)^2}{2!}(2.2103) + \frac{(0.1)^3}{3!}(2.2103)$$

$y(0.2) = 1.24275$ //

2. Solve $\frac{dy}{dx} = y^2 + x^2$, $y(0) = 1$ by Taylor series method Find y at $x = 0.1, 0.2, 0.3, 0.4$

Soln:

Taylor series formula

$$y_1 = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + \cdots$$

Given, $x_0 = 0, \quad y_0 = 1$

$y' = y^2 + x^2$

$y_0' = y_0^2 + x_0^2 = 1 + 0 = 1$

$y'' = 2yy' + 2x$

$y''' = 2[yy'' + y'y' + 1]$

$y_0'' = 2y_0 y_0' + 2x_0 = 2(1)(1) + 2(0) = 2$

$y_0''' = 2[y_0 y_0'' + y_0'^2] + \boxed{2} = 2[(1)(2) + (1)^2] + \boxed{2}$

$= 2[2+1] + \boxed{2}$

$= 6 + 2$

$= 8$

$$\therefore y = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \cdots$$

$$y = 1 + \frac{(x-0)}{1!}(1) + \frac{(x-0)^2}{2!}(2) + \frac{(x-0)^3}{6}(8) + \cdots$$

$$y = 1 + x + x^2 + \frac{4x^3}{3} + \cdots$$

$$y(0.1) = 1 + (0.1) + (0.1)^2 + \tfrac{4}{3}(0.1)^3 + \cdots = 1.1113$$

$$y(0.2) = 1 + (0.2) + (0.2)^2 + \tfrac{4}{3}(0.2)^3 + \cdots = 1.25066$$

$$y(0.3) = 1 + (0.3) + (0.3)^2 + \tfrac{4}{3}(0.3)^3 + \cdots = 1.426$$

$$y(0.4) = 1 + (0.4) + (0.4)^2 + \tfrac{4}{3}(0.4)^3 + \cdots = 1.64533$$

Home work:-

① using Taylor series method find y at $x = 0.1$ if $\frac{dy}{dx} = x^2 y - 1$, $y(0) = 1$.

② using Taylor series method find $y(1,1)$ given $y' = x + y$, $y(1) = 0$.

③ using Taylor series method with the first five terms in the expansion find $y(0.1)$ correct to three decimal places given that $\frac{dy}{dx} = e^x - y^2$, $y(0) = 1$

Soln:-

$$\frac{dy}{dx} = y' = e^x - y^2 \qquad x_0 = 0,\ y_0 = 1$$

$$y' = e^x - y^2 \qquad y_0' = e^{x_0} - y_0^2 = 1 - 1 = 0$$

$$y'' = e^x - 2yy' \qquad y_0'' = e^{x_0} - 2y_0 y_0' = 1 - 2(0) = 1$$

$$y''' = e^x - 2[yy'' + y'y'] \qquad y_0''' = e^{x_0} - 2[y_0 y_0'' + (y_0')^2]$$

$$= 1 - 2[1(1) + 0] = 1 - 2 = -1$$

$$\therefore y = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \cdots$$

$$= 1 + \frac{(x-0)}{1!}(0) + \frac{(x-0)^2}{2!}(1) + \frac{(x-0)^3}{3!}(-1)$$

$$= 1 + 0 + \frac{x^2}{2} + \frac{x^3}{6}(-1)$$

$$y = 1 + \frac{x^2}{2} - \frac{x^3}{6}$$

$$\therefore y(0.1) = 1 + \frac{(0.1)^2}{2} - \frac{(0.1)^3}{3!} = 1.004833 = 1.005$$

④ By means of Taylor series expansion, find y at $x = 0.1, 0.2$ correct to three significant digits given $\frac{dy}{dx} - 2y = 3e^x$, $y(0) = 0$.

Soln:

Here $x_0 = 0$, $y_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$ $h = 0.1$

$y' = 2y + 3e^x$ $\qquad y_0' = 2y_0 + 3e^{x_0} = 3$

$y'' = 2y' + 3e^x$ $\qquad y_0'' = 2y_0' + 3e^{x_0} = 9$

$y''' = 2y'' + 3e^x$ $\qquad y_0''' = 18 + 3 = 21$

$y^{iv} = 2y''' + 3e^x$ $\qquad y_0^{iv} = 42 + 3 = 45$

$$\therefore y_1 = y_0 + \frac{h}{1} y_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{6} y_0''' + \frac{h^4}{24} y_0^{iv} + \cdots$$

$$y(0.1) = y_1 = 0 + (0.1)(3) + \frac{(0.01)}{2}(9) + \frac{(0.001)}{6}(21) + \frac{(0.0001)}{24}(45) + \cdots$$

$$= 0.3 + 0.045 + 0.0035 + 0.0001875 + \cdots$$

$$= 0.3486875$$

$$= 0.349 \text{ (three decimals)}$$

$y_1' = 2y_1 + 3e^{x_1} = 0.3486875 \times 2 + 3e^{0.1} = 4.012887$

$y_1'' = 2y_1' + 3e^{x_1} = 11.025774$

$y_1''' = 2y_1'' + 3e^{x_1} = 25.3670608.$

$$y_2 = y(0.2) = y_1 + \frac{h}{1} y_1' + \frac{h^2}{2} y_1'' + \cdots$$

$$= 0.3486875 + (0.1)(4.012887) + \frac{0.01}{2}(11.341286) + \frac{(0.001)}{6}(25.99808) + \cdots$$

$$= 0.8110156 = 0.811 \text{ (Correct to three decimal places)}$$

The exact value of $y(0.1) = 0.3486955$ and $y(0.2) = 0.8112658$

<u>Taylor series method for Simultaneous First order Differential equations</u>

① Solve the system of equations $\frac{dy}{dx} = z - x^2$ $\frac{dz}{dx} = y + x$ with $y(0) = 1$, $z(0) = 1$ by taking $h = 0.1$ to get $y(0.1)$ and $z(0.1)$. Here y and z are dependent variables and x is independent.

Soln:-

$x_0 = 0 \quad y_0 = 1 \quad z_0 = 1$

$y' = z - x^2 \qquad y_0' = z_0 - x_0^2 = 1 - 0 = 1$

$y'' = z' - 2x \qquad y_0'' = z_0' - 2x_0 = 1 - 2(0) = 1$

$y''' = z'' - 2(1) \qquad y_0''' = z_0'' - 2 = 2 - 2 = 0$

$z' = y + x \qquad z_0' = y_0 + x_0 = 1 + 0 = 1$

$z'' = y' + 1 \qquad z_0'' = y_0' + 1 = 1 + 1 = 2$

$z''' = y'' \qquad z_0''' = y_0'' = 1$

$$\therefore y = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \cdots$$

$$y = 1 + \frac{(x-0)}{1!}(1) + \frac{(x-0)^2}{2}(1) + \frac{(x-0)^3}{3!}(0) + \cdots$$

$$y = 1 + x + x^2/2$$

$$y(0.1) = 1 + (0.1) + \frac{(0.1)^2}{2} = 1.105$$

$$z = z_0 + \frac{(x-x_0)}{1!} z_0' + \frac{(x-x_0)^2}{2!} z_0'' + \frac{(x-x_0)^3}{3!} z_0''' + \cdots$$

$$z = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(1) \cdots$$

$$z(0.1) = 1 + (0.1) + (0.1)^2 + \frac{(0.1)^3}{6}$$

$$z(0.1) = 1.110166 \text{ //Ans}$$

HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

① By Taylor's series method find $y(0.1)$ given that $y'' = y + xy'$, $y(0) = 1$, $y'(0) = 0$

Soln:- Here $x_0 = 0 \quad y_0 = 1 \quad y_0' = 0$

Given $y'' = y + xy'$

$y''' = y' + xy'' + y'$

$= 2y' + xy''$

$y_0'' = y_0 + x_0 y_0' = 1 + (0)(0) = 1$

$y_0''' = 2y_0' + x_0 y_0'' = 2(0) + (0)(1) = 0$

$y_0^{iv} = 3y_0'' + x_0 y_0''' = 3(1) + (0)(0) = 3$

$$y^{iv} = 2y'' + xy''' + y''$$

$$= 3y'' + xy'''$$

$$\therefore y(x) = y_0 + xy_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \frac{x^4}{4!} y^{iv}$$

$$= 1 + 0 + \frac{x^2}{2}(1) + 0 + \frac{x^4}{24}(3) + \cdots$$

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{8}$$

$$\therefore y(0.1) = 1 + \frac{(0.1)^2}{2} + \frac{(0.1)^4}{8}$$

$$= 1.0050$$

② Evaluate the values of $y(0.1)$ and $y(0.2)$ given $y'' - x(y')^2 + y^2 = 0$, $y(0) = 1$, $y'(0) = 0$ by using Taylor series method.

Soln:-

Given $y'' = x(y')^2 - y^2$, $x_0 = 0$ $y_0 = 1$ $y_0' = 0$

$$y''' = x\,2y'\,y'' + (y')^2 - 2yy'$$

$$= 2xy'y'' + (y')^2 - 2yy'$$

$$= y'\left[2xy'' + y' - 2y\right]$$

$$y_0'' = x_0(y_0')^2 - y_0^2$$

$$= (0)(0)^2 - (1)^2 = -1$$

$$y_0''' = 2x_0 y_0' y_0'' + (y_0')^2 - 2y_0 y_0'$$

$$= 2(0)(0)(-1) + (0)^2 - 2(1)(0)$$

$$= 0 + 0 + 0$$

$$= 0$$

$$y^{iv} = y'\left[2(xy''' + y'') + y'' - 2y'\right] + (2xy'' + y' - 2y)\,y''$$

$$= 2xy'y''' + 2y'y'' - 2(y')^2 + 2x(y'')^2 + y'y'' - 2yy'$$

$$= 2xy'y''' + 4y'y'' + 2x(y'')^2 - 2(y')^2 - 2yy''$$

$$y_0^{(iv)} = 2x_0 y_0' y_0''' + 4y_0' y_0'' + 2x_0(y_0'')^2 - 2(y_0')^2 - 2y_0 y_0''$$

$$= 0 + 0 + 0 - 2(0) - 2(1)(-1)$$

$$= 2$$

$$\therefore y(x) = y_0 + xy_0' + \frac{x^2}{2} y_0'' + \frac{x^3}{3!} y_0''' + \frac{x^4}{4!} y_0^{iv} + \cdots$$

$$= 1 + (0) + \frac{x^2}{2}(-1) + \frac{x^4}{24}(2) + \cdots$$

$$= 1 - \frac{x^2}{2} + \frac{1}{12}x^4 + \cdots$$

$$y(0.1) = 1 - \frac{(0.1)^2}{2} + \frac{(0.1)^4}{12} + \cdots = 1 - 0.005 + 0.000008.$$

$$= 0.995008 \quad = 0.995 \text{ (Correct to 3 decimal places)}$$

Euler and modified Euler method.

$$\boxed{\begin{aligned} y_1 &= y_0 + h f(x_0, y_0) \\ y_2 &= y_1 + h f(x_1, y_1) \\ &\vdots \\ y_{n+1} &= y_n + h f(x_n, y_n), \quad n = 0, 1, 2 \ldots \end{aligned}}$$

① using Euler's method find $y(0.2)$ and $y(0.4)$, $y(0.6)$ from $\frac{dy}{dx} = x+y$, $y(0)=1$ with $h = 0.2$

Soln:

$f(x, y) = x + y$, $\quad x_0 = 0$, $y_0 = 1$ $\quad x_1 = 0.2$, $x_2 = 0.4$

By Euler algorithm,

$$y_1 = y_0 + h f(x_0, y_0)$$

$$= 1 + (0.2)[x_0 + y_0]$$

$$= 1 + (0.2)(0+1)$$

$$y_1 = 1.2$$

i.e. $y(0.2) = 1.2$

$$y_2 = y_1 + h f(x_1, y_1)$$

$$= 1.2 + 0.2[x_1 + y_1]$$

$$= 1.2 + (0.2)[0.2 + 1.2]$$

$$= 1.2 + (0.2)[1.4]$$

$$= 1.2 + 0.28$$

$$y_2 = 1.48$$

$$y(0.4) = 1.48$$

$$y_3 = y_2 + h f(x_2, y_2)$$

$$= 1.48 + (0.2)[x_2 + y_2]$$

$$= 1.48 + (0.2)[0.4 + 1.48]$$

$$= 1.48 + 0.376$$

$$y(0.6) = 1.856$$

② Using Euler's method find $y(0.3)$ of $y(x)$ satisfies the initial value problem. $\frac{dy}{dx} = \frac{1}{2}(x^2+1)y^2$, $y(0.2) = 1.1114$.

Soln:

Given $f(x,y) = \frac{1}{2}(x^2+1)y^2$, $\quad x_0 = 0.2$, $y_0 = 1.1114$, $x_1 = 0.3$

By Euler,

$$y_1 = y_0 + h\,f(x_0, y_0)$$

$$= y_0 + h\left[\tfrac{1}{2}(x_0^2+1)y_0^2\right]$$

$$= 1.1114 + \frac{(0.1)}{2}\left[((0.2)^2+1)(1.1114)^2\right]$$

$$= 1.1114 + \frac{0.1}{2}\left[1.2846184\right]$$

$$= 1.1114 + 0.0642$$

$$y(0.3) = 1.1756$$

③ Compute y at $x = 0.25$ by modified Euler method given $y' = 2xy$, $y(0) = 1$.

Soln:

Given $f(x,y) = 2xy$, $x_0 = 0$, $y_0 = 1$, $h = 0.25$, $x_1 = 0.25$

By modified Euler method,

$$y_{n+1} = y_n + h\,f\left(x_n + \tfrac{1}{2}h,\; y_n + \tfrac{1}{2}h\,f(x_n, y_n)\right)$$

$$y_1 = y_0 + h\,f\left(0 + \frac{0.25}{2},\; 1 + \frac{0.25}{2}(2x_0 y_0)\right)$$

$$= 1 + (0.25)\left[f(0.125, 1 + (0.25)(0)(1))\right]$$

$$= 1 + 0.25\left[2(0.125)(1)\right]$$

$$= 1 + 0.00625$$

$$= 1.00625$$

④ Using modified Euler's method, compute $y(0.1)$ with $h = 0.1$ from $y' = y - \frac{2x}{y}$, $y(0) = 1$.

$x_1 = 0.1$ $\quad h = 0.1$

Soln: Given $f(x,y) = y - \frac{2x}{y}$, $x_0 = 0$ $y_0 = 1$

By modified Euler method.

$$y_1 = y_0 + h\left[f\left[x_0 + \tfrac{h}{2},\; y_0 + \tfrac{h}{2} f(x_0, y_0)\right]\right]$$

$$= 1 + (0.1)\left[f\left(0 + \frac{0.1}{2}\right),\; 1 + \frac{0.1}{2}\left[y_0 - \frac{2x_0}{y_0}\right]\right]$$

$= 1 + (0.1)\left[f[(0.05), 1 + (0.05)[1-0]]\right]$

$= 1 + (0.1)\left[1.05 - \frac{2(0.05)}{1.05}\right]$

$= 1 + (0.1)[1.05 - 0.0952]$

$= 1 + (0.1)[0.9548]$

$= 1 + 0.09548$

$= 1.09548$ //Ans

(5) using modified Euler's method, find $y(0.1)$ if $\frac{dy}{dx} = y - x^2 + 1$, $y(0) = 1$

Soln:

Given $f(x,y) = y - x^2 + 1$, $x_0 = 0$, $y_0 = 1$ $h = 0.1$, $x_1 = 0.1$

By modified Euler method.

$$y_{n+1} = y_n + h\, f\left[x_n + \frac{h}{2},\ y_n + \frac{h}{2} f(x_n, y_n)\right]$$

$$y_1 = y_0 + h f\left[x_0 + \frac{h}{2},\ y_0 + \frac{h}{2} f(x_0, y_0)\right]$$

$= 0.5 + (0.2)\, f\left[0 + \frac{0.2}{2}, 0.5 + \frac{0.2}{2} f(0, 0.5)\right]$

$= 0.5 + (0.2)\, f[0.1, 0.5 + (0.1)[0.5 - 0 + 1]]$

$= 0.5 + (0.2)\, f[0.1, 0.5 + (0.1)[1.5]]$

$= 0.5 + (0.2)\, f[0.1, 0.5 + 0.15]$

$= 0.5 + (0.2)\, f[0.1, 0.65]$

$= 0.5 + (0.2)\, f[0.1, 0.65]$

$= 0.5 + (0.2)[0.65 - (0.1)^2 + 1]$

$= 0.5 + (0.2)[0.65 - 0.01 + 1]$

$= 0.5 + (0.2)[1.64]$

$= 0.5 + 0.328$

$y_1 = 0.828$ //Ans.

RUNGE - KUTTA METHOD : (FOURTHORDER)

$$y_1 = y_0 + \Delta y$$

where $K_1 = h f(x_0, y_0)$

$$K_2 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_1}{2}\right)$$

$$K_3 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right)$$

$$K_4 = h f\left(x_0 + h,\ y_0 + K_3\right)$$

and $\Delta y = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

① Given $\frac{dy}{dx} = x^3 + y$, $y(0) = 2$. Compute $y(0.2)$, $y(0.4)$ and $y(0.6)$ by Runge-kutta method of fourth order.

Soln:

Given $y' = x^3 + y = f(x, y)$, $x_0 = 0$, $y_0 = 2$, $x_1 = 0.2$

Fourth order RK algorithm

$$K_1 = h f(x, y)$$

$$K_2 = h f\left[x + h/2,\ y + \frac{K_1}{2}\right]$$

$$K_3 = h f\left[x + \frac{h}{2},\ y + \frac{K_2}{2}\right]$$

$$K_4 = h f\left[x + h,\ y + K_3\right]$$

$$\Delta y = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$y(x+h) = y(x) + \Delta y$$

First method

$$K_1 = h f(x_0, y_0)$$

$$= 0.2[x_0^3 + y_0]$$

$$= (0.2)[0 + 2]$$

$$= 0.4$$

$$K_2 = h f\left[x_0 + h/2,\ y_0 + \frac{K_1}{2}\right]$$

$$= (0.2) f\left[0 + \frac{0.2}{2},\ 2 + \frac{0.4}{2}\right]$$

$$= (0.2) f[0 + 0.1,\ 2 + 0.2]$$

$$= (0.2) f[0.1,\ 2.2]$$

$= (0.2)\left[(0.1)^3 + 2.2\right]$

$= (0.2)\left[2.201\right]$

$= 0.4402.$

$K_3 = hf\left[x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right]$

$= (0.2)\, f\left[0 + \frac{0.2}{2},\ 2 + \frac{0.4402}{2}\right]$

$= (0.2)\, f\left[0.1,\ 2.2201\right]$

$= (0.2)\left[(0.1)^3 + 2.2201\right]$

$= (0.2)\left[2.2211\right]$

$= 0.44422$

$K_4 = hf\left[x_0 + h,\ y_0 + K_3\right]$

$= (0.2)\, f\left[0 + 0.2,\ 2 + 0.44422\right]$

$= (0.2)\, f\left[0.2,\ 2.44422\right]$

$= (0.2)\left[(0.2)^3 + 2.44422\right]$

$= (0.2)\left[2.45222\right]$

$= 0.490444$

$\Delta y = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

$= \frac{1}{6}\left[0.4 + 2(0.4402) + 2(0.44422) + 0.490444\right]$

$= \frac{1}{6}\left[2.65928\right]$

$\Delta y = 0.44321$

$y(0.2) = y_1 = y_0 + \Delta y = 2 + 0.44321 = 2.44321 = 2.443$

(Correct to 3 decimals)

Again apply R.K method (Second integral)

$K_1 = hf(x_1, y_1) = (0.2)\, f\left[0.2,\ 2.443\right]$

$= (0.2)\left[(0.2)^3 + 2.443\right]$

$= (0.2)\ [2.451]$

$K_1 = 0.4902$

$K_2 = hf\left[x_1 + h/2,\ y_1 + \frac{K_1}{2}\right]$

$= (0.2)\ f\left[0.2 + \frac{0.2}{2},\ 2.443 + \frac{0.4902}{2}\right]$

$= (0.2)\ f\ [0.3,\ 2.6881]$

$= (0.2)\ [(0.3)^3 + 2.6881]$

$= (0.2)\ [2.7151]$

$K_2 = 0.5430$

$K_3 = hf\left[x_1 + \frac{h}{2},\ y_1 + \frac{K_2}{2}\right]$

$= (0.2)\ f\left[0.2 + \frac{0.2}{2},\ 2.443 + \frac{0.543}{2}\right]$

$= (0.2)\ f\ [0.3,\ 2.7145]$

$= (0.2)\ [(0.3)^3 + 2.7145]$

$= (0.2)\ [2.7415]$

$= 0.5483$

$K_4 = hf\ [x_1 + h,\ y_1 + K_3]$

$= (0.2)\ f\ [0.2 + 0.2,\ 2.443 + 0.5483]$

$= (0.2)\ f\ [0.4,\ 2.9913]$

$= (0.2)\ [(0.4)^3 + 2.9913]$

$= (0.2)\ (3.0553)$

$= 0.6111$

$\Delta y = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

$= \frac{1}{6}\left[0.4902 + 2.(0.543) + 2(0.5483) + 0.6111\right]$

$= \frac{1}{6}\ [3.2839]$

$= 0.5473$

$y(0.4) = y_2 = y_1 + \Delta y = 2.443 + 0.5473 = 2.99$ (correct to 3 decimal)

Again apply R.K method (third interval)

Here $x_2 = 0.4 \quad y_2 = 2.99$

$K_1 = hf(x_2, y_2)$

$= (0.2) f(0.4, 2.99)$

$= (0.2)[(0.4)^3 + 2.99] = (0.2)[3.054] = 0.6108$

$K_2 = hf\left[x_2 + \frac{h}{2}, y_2 + \frac{K_1}{2}\right]$

$= (0.2) f\left[(0.4) + \frac{0.2}{2}, 2.99 + \frac{0.6108}{2}\right]$

$= (0.2) f[0.5, 3.2954]$

$= (0.2)[(0.5)^2 + 3.2954] = (0.2)(3.4204) = 0.6841$

$K_3 = hf\left[x_2 + \frac{h}{2}, y_2 + \frac{K_2}{2}\right]$

$= (0.2) f\left[0.4 + \frac{0.2}{2}, 2.99 + \frac{0.6841}{2}\right]$

$= (0.2) f[0.5, 3.3321]$

$= (0.2)[(0.5)^3 + 3.3321]$

$= (0.2)[3.4571] = 0.6914$

$K_4 = hf[x_2 + h, y_2 + K_3]$

$= (0.2) f[0.4 + 0.2, 2.99 + 0.6914]$

$= (0.2) f[0.6, 3.6814]$

$= (0.2)[(0.6)^3 + 3.6814]$

$= (0.2)[3.8974]$

$= 0.7795$

$\Delta y = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= \frac{1}{6}[0.6108 + 2(0.6841) + 2(0.6914) + 0.7795]$

$= \frac{1}{6}[4.1413]$

$\Delta y = 0.6902$

$y(0.6) = y_3 = y_2 + \Delta y = 2.99 + 0.6902 = 3.68$

x	0	0.2	0.4	0.6
y	2	2.443	2.99	3.68

② using R.K method of 4th order, solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0)=1$ at $x=0.2$

Soln:

Given $y' = f(x,y) = \frac{y^2-x^2}{y^2+x^2}$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.2$, $h = 0.2$

$K_1 = h f(x_0, y_0) = (0.2)\left[\frac{y_0^2 - x_0^2}{y_0^2 + x_0^2}\right] = (0.2)\left[\frac{1-0}{1+0}\right] = 0.2$

$K_2 = h f\left[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right]$

$= (0.2) f\left[0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right]$

$= (0.2) f[0.1, 1.1] = (0.2)\left[\frac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2}\right]$

$= (0.2)\left[\frac{1.2}{1.22}\right]$

$= (0.2)[0.9836]$

$= 0.19672$

$K_3 = h f\left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right]$

$= (0.2) f\left[0 + \frac{0.2}{2}, 1 + \frac{0.19672}{2}\right]$

$= (0.2) f[0.1, 1.0983606]$

$= (0.2)\left[\frac{(1.0983606)^2 - (0.1)^2}{(1.0983606)^2 + (0.1)^2}\right]$

$= 0.1967$

$K_4 = h f(x_0 + h, y_0 + K_3)$

$= (0.2) f(0.2, 1.1967)$

$= (0.2)\left[\frac{(1.1967)^2 - (0.2)^2}{(1.1967)^2 + (0.2)^2}\right]$

$= 0.1891$

$$\Delta y = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$= \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891]$$

$$= 0.19598$$

$$y(0.2) = y_1 = y_0 + \Delta y = 1 + 0.19598 = 1.19598$$

RUNGE - KUTTA METHOD FOR SECOND ORDER DIFFERENTIAL EQUATIONS

① consider the second order initial value problem $y'' - 2y' + 2y = e^{2t}\sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$ using Fourth order R.K method, find $y(0.2)$

Soln:

let $t = x$

$y'' = 2y' - 2y + e^{2x}\sin x, \quad y(0) = -0.4 \quad y'(0) = -0.6 \quad h = 0.2$

Setting $y' = z$ the equation becomes

$$z' = 2z - 2y + e^{2x}\sin x.$$

$f_1(x, y, z) = \frac{dy}{dx} = z, \quad f_2(x, y, z) = \frac{dz}{dx} = 2z - 2y + e^{2x}\sin x$

Given: $y_0 = -0.4$, $z_0 = y_0' = -0.6$, $x_0 = 0$

$K_1 = hf_1(x_0, y_0, z_0)$
$= (0.2)[z_0]$
$= (0.2)[-0.6]$
$= -0.12$

$l_1 = hf_2(x_0, y_0, z_0)$
$= (0.2)[2z_0 - 2y_0 + e^{2x_0}\sin x_0]$
$= (0.2)[2(-0.6) - 2(-0.4) + e^{2(0)}\sin(0)]$
$= (0.2)[-0.12 + 0.8]$
$\boxed{l_1 = 0.136}$

$K_2 = hf_1\left[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right]$

$= (0.2) f_1\left[0 + \frac{0.2}{2}, -0.4 - \frac{0.12}{2}, -0.6 + \frac{0.136}{2}\right]$

$= (0.2) f_1[0.1, -0.46, -0.532]$

$= (0.2)[-0.532]$

$= -0.1064$

$$l_2 = h f_2\left[x_0+\frac{h}{2},\ y_0+\frac{k_1}{2},\ z_0+\frac{l_1}{2}\right]$$

$$=(0.2) f_2\left[0+\frac{0.2}{2}, -0.4-\frac{0.12}{2}, -0.6+\frac{0.136}{2}\right]$$

$$=(0.2) f_2[0.1, -0.46, -0.532]$$

$$=(0.2)\left[2(-0.532)-2(-0.46)\right]+e^{2(0.1)}\sin(0.1)$$

$$=(0.2)[-1.064+0.92+0.1294]$$

$$=-0.00292$$

$$k_3 = h f_1\left[x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right]$$

$$=(0.2) f_1\left[0+\frac{0.2}{2}, -0.4-\frac{0.1064}{2}, -0.6-\frac{0.00292}{2}\right]$$

$$=0.2 f_1[0.1, -04532, -0.60146]$$

$$=0.2[-0.60146]$$

$$=-0.1203$$

$$l_3 = h f_2\left[x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right]$$

$$=(0.2) f_2\left[0+\frac{0.2}{2}, -0.4-\frac{0.1064}{2}, -0.6-\frac{0.00292}{2}\right]$$

$$=(0.2) f_2[0.1, -0.4532, -0.60146]$$

$$=(0.2)\left[2(-0.60146)-2(-0.4532)+e^{2(0.1)}\sin(0.1)\right]$$

$$=(0.2)[-1.20292+0.9064+0.12194]$$

$$=-0.0105$$

$$k_4 = h f_1[x_0+h, y_0+k_3, z_0+l_3]$$

$$=(0.2) f_1[0+0.2, -0.4-0.1203, -0.6-0.0105]$$

$$=(0.2) f_1[0.2, -0.5203, -0.6105]$$

$$=(0.2)[-0.6105]$$

$$=-0.1221$$

$$l_4 = h f_2[x_0+h, y_0+k_3, z_0+l_3]$$

$$=(0.2) f_2[0+0.2, -0.4-0.1203, -0.6-0.0105]$$

$$=(0.2)\left[2(-0.6105)-2(-0.5203)+e^{2(0.2)}\sin(0.2)\right]$$

$$=(0.2)[-1.221+1.0406+0.29638]$$

$$=0.0825$$

$$\Delta y = \frac{1}{6}[k_1+2k_2+2k_3+k_4]$$

$$=\frac{1}{6}[-0.12+2(-0.1064)+2(-0.1203)+(-0.1221)]$$

$$=-\frac{1}{6}[0.12+2(0.1064)+2(0.1203)+0.1221]$$

$$=-0.1159$$

$$y_1 = y_0+\Delta y = -0.4-0.1159$$

$$=-0.5159$$

$$y(0.2) = -0.5159$$

$$\Delta z = \frac{1}{6}[l_1+2l_2+2l_3+l_4]$$

$$=\frac{1}{6}[0.136+2(-0.00292)+2(-0.0105)+0.0825]$$

$$=\frac{1}{6}[0.136-2(0.00292)-2(0.0105)+0.0825]$$

$$=0.03194$$

$$z_1 = z_0+\Delta z = -0.6+0.3194 = -0.2806$$

MILNE'S PREDICTOR AND CORRECTOR METHODS

milne's predictor corrector formulae

① predictor

$$y_{n+1},\ P = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

② Corrector

$$y_{n+1},\ C = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

① using milne's method find $y(4.4)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ and $y(4.3) = 1.0143$

Soln:

$y' = \frac{2 - y^2}{5x}$, $x_0 = 4$ $x_1 = 4.1$, $x_2 = 4.2$, $x_3 = 4.3$

$x_4 = 4.4$ $y_0 = 1$ $y_1 = 1.0049$, $y_2 = 1.0097$, $y_3 = 1.0143$

$$y_1' = \frac{2 - y_1^2}{5x_1} = \frac{2 - (1.0049)^2}{5(4.1)} = 0.0493$$

$$y_2' = \frac{2 - y_2^2}{5x_2} = \frac{2 - (1.0097)^2}{5(4.2)} = 0.0467$$

$$y_3' = \frac{2 - y_3^2}{5x_3} = \frac{2 - (1.0143)^2}{5(4.3)} = 0.0452$$

By milne's predictor formula

$$y_4,\ P = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

$$= 1 + \frac{4(0.1)}{3}\left[2(0.0493) - 0.0467 + 2(0.0452)\right]$$

$$= 1.01897$$

$$y_4' = \frac{2 - y_4^2}{5(x_4)} = \frac{2 - (1.01897)^2}{5(4.4)} = 0.0437$$

using

$$y_4,\ C = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

$$= 1.0097 + \frac{0.1}{3}\left[0.0467 + 4(0.0452) + 0.0437\right]$$

$$y_4, C = 1.01874$$

② Solve $y' = x - y^2$, $0 \le x \le 1$, $y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$ by milne's method to find $y(0.8)$ and $y(1)$.

Soln:-

Here
$x_0 = 0$ $\quad y_0 = 0$

$x_1 = 0.2$ $\quad y_1 = 0.02$ $\qquad h = 0.2$

$x_2 = 0.4$ $\quad y_2 = 0.0795$

$x_3 = 0.6$ $\quad y_3 = 0.1762$

$x_4 = 0.8$ $\quad y_4 = ?$

$x_5 = 1$ $\quad y_5 = ?$

$$y' = f(x, y) = x - y^2 \quad \text{—— ①}$$

By milne's predictor formula

$$y_{n+1, P} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n]$$

$$y_{4, P} = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3] \quad \text{—— ②}$$

from (1) $y' = x - y^2$

$$y'_1 = x_1 - y_1^2 = 0.2 - (0.02)^2 = 0.1996$$

$$y'_2 = x_2 - y_2^2 = 0.4 - (0.0795)^2 = 0.3937$$

$$y'_3 = x_3 - y_3^2 = 0.6 - (0.1762)^2 = 0.5690$$

$$y_{4, P} = 0 + \frac{4(0.2)}{3}[2(0.1996) - (0.3937) + 2(0.5690)]$$

$$= \frac{0.8}{3}[1.1435] = 0.3049$$

$$y'_4 = x_4 - y_4^2$$

$$= 0.8 - (0.3049)^2$$

$$= 0.707$$

$$y_{4, C} = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4]$$

$$= 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5690) + 0.707]$$

$$= 0.0795 + \frac{0.2}{3}[3.3767]$$

$$= 0.3046$$

Corrected value of y at $x = 0.8$ is 0.3046

To find $y(1)$

$$y_{5,P} = y_1 + \frac{4h}{3}[2y_2' - y_3' + 2y_4']$$

$$= 0.02 + \frac{4(0.2)}{3}[2(0.3937) - (0.5690) + 2(0.707)]$$

$$= 0.02 + \frac{4(0.2)}{3}[1.6324]$$

$$= 0.02 + \frac{1.30592}{3} = 0.4553$$

$$y_{5,P}' = x_5 - y_5^2 = 1 - (0.4533)^2 = 0.7327$$

$$y_{5,C} = y_3 + \frac{h}{3}[y_3' + 4y_4' + y_5']$$

$$= (0.1762) + \frac{0.2}{3}[0.569 + 4(0.707) + 0.7327]$$

$$= 0.1762 + \frac{0.2}{3}[4.1297]$$

$$= 0.4515$$

Corrected value of y at $x = 1$ is 0.4515.

(3) Given $\frac{dy}{dx} = x^3 + y$, $y(0) = 2$

The values of $y(0.2) = 2.073$, $y(0.4) = 2.452$, and $y(0.6) = 3.023$ are got by R-K method of fourth order. Find $y(0.8)$ by Milne's predictor-corrector method taking $h = 0.2$

Soln:

Here

$x_0 = 0$	$y_0 = 2$
$x_1 = 0.2$	$y_1 = 2.073$
$x_2 = 0.4$	$y_2 = 2.452$
$x_3 = 0.6$	$y_3 = 3.023$

$$y' = f(x, y) = x^3 + y \quad \text{---} (1)$$

By Milne's predictor formula

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n]$$

$$y_{4,P} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3] \quad \text{---} (2)$$

From (1) $y' = x^3 + y$

$y_1' = x_1^3 + y_1 = (0.2)^3 + 2.073 = 2.081$

$y_2' = x_2^3 + y_2 = (0.4)^3 + 2.452 = 2.516$

$y_3' = x_3^3 + y_3 = (0.6)^3 + 3.023 = 3.239$

$(2) \Rightarrow y_4, P = 2 + \frac{4(0.2)}{3}[2(2.081) - 2.516 + 2(3.239)]$

$= 2 + \frac{0.8}{3}[8.124]$

$= 2 + 2.1664$

$= 4.1664$

using milne's corrector formula.

$y_{n+1}, C = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$

$y_4, C = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$ —— (3)

$y_1' = x_1^3 + y_1$

$y_2' = x_2^3 + y_2 = 2.516$

$y_3' = x_3^3 + y_3 = 3.239$

$y_4' = x_4^3 + y_4 = (0.8)^3 + 4.1664 = 4.6784$

$(3) \Rightarrow y_4, C = 2.452 + \frac{0.2}{3}[2.516 + 4(3.239) + 4.6784]$

$= 2.452 + \frac{0.2}{3}[20.1504]$

$= 3.79536$

Corrected value of y at $x = 0.8$ is 3.79536.

ADAM'S PREDICTOR AND CORRECTOR METHODS

Predictor :-

$$y_{n+1}, P = y_n + \frac{h}{24}\left[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right]$$

Corrector :-

$$y_{n+1}, C = y_n + \frac{h}{24}\left[9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}'\right]$$

① Given $\frac{dy}{dx} = x^2(1+y)$ $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$ $y(1.3) = 1.979$, evaluate $y(1.4)$ by Adams - Bashforth method.

Soln:-

$x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$, $x_3 = 1.3$, $x_4 = 1.4$

$y_0 = 1$, $y_1 = 1.233$, $y_2 = 1.548$, $y_3 = 1.979$, $y_4 = ?$

By Adam's method

predictor:-

$$y_{n+1}, P = y_n + \frac{h}{24}\left(55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right]$$

$$y_4, P = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

Here $y_0' = x_0^2(1+y_0) = (1)^2(1+1) = 2$

$y_1' = x_1^2(1+y_1) = (1.1)^2(1+1.233) = 2.70193$

$y_2' = x_2^2(1+y_2) = (1.2)^2[1+1.548] = 3.66912$

$y_3' = x_3^2(1+y_3) = (1.3)^2 \cdot [1+1.979] = 5.0345$

$(1) \Rightarrow y_4, P = 1.979 + \frac{0.1}{24}\left[55(5.0345) - 59(3.60912) + 37(2.70193) - 9(2)\right]$

$= 1.979 + \frac{0.1}{24}\left[276.8975 - 212.93808 + 99.91741 - 18\right]$

$= 1.979 + \frac{0.1}{24}\left[145.93083\right]$

$= 1.979 + 0.6080451$

$= 2.5870451$

$= 2.5871$ [correct to four decimal places] corrector method:

$$y_{n+1, C} = y_n + \frac{h}{24}\left[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}\right]$$

$$y_{4, C} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

$$y_4' = x_4^2(1+y_4) = (1.4)^2\left[1 + 2.5871\right] = 7.030716$$

$$(2) \Rightarrow y_{4, C} = 1.979 + \frac{0.1}{24}\left[9(7.030716) + 19(5.0345) - 5(3.60912) + 2.70193\right]$$

$$= 1.979 + \frac{0.1}{24}\left[63.276444 + 95.6555 - 18.0456 + 2.70193\right]$$

$$= 1.979 + \frac{0.1}{24}\left[143.58827\right]$$

$$= 1.979 + 0.592844$$

$$= 2.5772844$$

$$= 2.5773 \text{ [correct to four decimal places]}$$

②. using Adam's Bashforth method find $y(4.4)$ given $5xy' + y^2 = 2$, $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ and $y(4.3) = 1.0143$

Soln:

Given $y' = \dfrac{2-y^2}{5x}$, let $h = 0.1$

Given $x_0 = 4$, $y_0 = 1$, $x_1 = 4.1$, $y_1 = 1.0049$,

$x_2 = 4.2$ $y_2 = 1.0097$, $x_3 = 4.3$ $y_3 = 1.0143$

Adams predictor formulla is

$$y_{n+1, P} = y_n + \frac{h}{24}\left[55y_n' - 59y'_{n-1} - 37y'_{n-2} - 9y'_{n-3}\right]$$

putting $n=3$ we have

$$y_{4, P} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$y_0' = (y')(x_0, y_0) = \frac{2-y_0^2}{5x_0} = 0.05$$

$$y_1' = (y')(x_1, y_1) = \frac{2-y_1^2}{5x_1} = 0.0483$$

$$y_2' = (y')(x_2, y_2) = \frac{2-y_2^2}{5x_2} = 0.0467$$

$$y_3' = (y')(x_3, y_3) = \frac{2-y_3^2}{5x_3} = 0.0452$$

using these values in (1) we get

$$y_{4, P} = 1.0143 + \frac{0.1}{24}\left[55(0.0452) - 59(0.0467) + 37(0.0483) - 9(0.05)\right]$$

$= 1.01413 + \frac{0.1}{24}[4.2731 - 3.2053] = 1.0186$

$y(4.4) = 1.0186$

Adam's corrector formula is

$y_{n+1, c} = y_n + \frac{h}{24}[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}]$

putting n=3 we get

Now $y'_4 = y'(x_4, y_4) = \frac{2 - y_4^2}{5x_4} = 0.0437$

∴ (2) Becomes

$y_{4, c} = 1.0143 + \frac{0.1}{24}[9(0.0437) + 19(0.0452) - 5(0.0467) + 0.0483]$

$= 1.0143 + \frac{0.1}{24} \times 1.0669 = 1.0187$

∴ $y(4.4) = 1.0187$ Ans