when $u = \frac{x - x_0}{h}$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \cdots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \cdots\right]$$

Derivative using Bessel's formula

$$y(x) = \frac{1}{2}[y_0 + y_1] + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{4}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{u(u-\frac{1}{2})(u-1)}{6}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{48}(\Delta^4 y_{-2} + \Delta^4 y_{-1})$$

$$y'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{4}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{(3u^2 - 3u + \frac{1}{2})}{6}\Delta^3 y_{-1} + \cdots\right]$$

Maxima and minimal of a tabulated function

For maxima or minima $\frac{dy}{dx} = 0$ Hence equating the right hand side of (1) to zero and retaining only upto third differences we obtain

$$\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{6}\Delta^3 y_0 = 0$$

(ie) $\left(\frac{1}{2}\Delta^3 y_0\right)u^2 + (\Delta^2 y_0 - \Delta^2 y_0)u + \left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{2}\Delta^3 y_0\right) = 0$

Substituting the values of $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ from the difference table, we solve this quadratic form. Then the corresponding values of x are given by $x = x_0 + uh$ at which y is maximum (or) minimum.

Problems:-

① Find $f'(3)$ and $f''(3)$ for one following data:

x:	3.0	3.2	3.4	3.6	3.8	4.0
$f(x)$:	-14	-10.032	-5.296	-0.256	6.672	14

Solution:-

Since we require $f'(3)$ and $f''(3)$ we use Newton's forward formula.

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
3.0	−14					
		3.968				
3.2	−10.032		0.768			
		4.736		−0.464		
3.4	−5.296		0.304		2.048	
		5.04		1.584		−5.12
3.6	−0.256		1.888		−3.072	
		6.928		−1.488		
3.8	6.672		0.4			
		7.328				
4.0	14					

By Newton's forward formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right]$$

Here $h = 0.2$

$$= \frac{1}{0.2}\left[3.968 - \frac{1}{2}(0.768) + \frac{1}{3}(-0.464) - \frac{1}{4}(2.048) + \frac{1}{5}(-5.12)\right]$$

$$= \frac{1}{0.2}\left[3.968 - 0.384 - 0.1547 - 0.512 - 1.024\right]$$

$$= \frac{1}{0.2}\left[1.8933\right]$$

$$= 9.4665$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

$$= \frac{1}{(0.2)^2}\left[0.768 - (-0.464) + \frac{11}{12}(2.048) - \frac{5}{6}(-5.12)\right]$$

$$= \frac{1}{0.04}\left[0.768 + 0.464 + 1.8773 + 4.267\right]$$

$$= \frac{1}{0.04}\left[7.3763\right]$$

$$= 184.4075$$

② Compute $f'(0)$ and $f''(4)$ from the data.

x	0	1	2	3	4
y	1	2.718	7.381	20.086	54.598

Solution:-

Since we require $f'(0.5)$ and $f''(3.5)$ we use Newton's forward formula and Newton's backward formula.

Difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		1.718			
1	2.718		2.945		
		4.663		5.097	
2	7.381				8.668
			8.042		
		12.705			
3	20.0826			13.765	
			21.807		
		34.512			
4	54.598				

By Newton's forward formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$= \frac{1}{1}\left[1.718 - \frac{1}{2}(2.945) + \frac{1}{3}(5.097) - \frac{1}{4}(8.668)\right]$$

$$= [1.718 - 1.4725 + 1.699 - 2.167]$$

$$= -0.2225$$

By Newton's backward difference formula

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \left(\frac{dy}{dx}\right)_{v=0} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right]$$

$$= \frac{1}{1}\left[(34.512) + \frac{1}{2}(21.807) + \frac{1}{3}(13.765) + \frac{1}{4}(8.668)\right]$$

$$= 34.512 + 10.9035 + 4.588 + 2.167$$

$$= 52.1705$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right]$$

$$= \frac{1}{1}\left[21.807 + 13.765 + \frac{11}{12}(8.668)\right]$$

$$= 21.807 + 13.765 + 7.9457$$

$$= 43.5177 \, m/s$$

③ Find the maximum and minimum value of y tabulated below.

x	-2	-1	0	1	2	3	4
y	2	-0.25	0	-0.25	2	15.75	56

<u>Solution:</u>

Newton's forward difference formula is

$$y(x) = \frac{1}{h}\left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots \right]$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \dots \right]$$

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
-2	2					
		-2.25				
-1	-0.25		2.5			
				-3		
		0.25			6	
			-0.5			
0	0			3		0
		$\boxed{-0.25}$			6	
1	-0.25		$\boxed{2.5}$			0
				$\boxed{9}$		
		2.25				
2	2		11.5		$\boxed{6}$	
		13.75				
				15		
3	15.75		26.5			
		40.25				
4	56					

Choosing $x_0 = 0$, $u = \frac{x-0}{1} = x$

$$\frac{dy}{dx} = \frac{1}{1}\left[-0.25 + \frac{(2x-1)}{2}(2.5) + \frac{3x^2-6x+2}{6}(9) + \frac{4x^3-18x^2+22x-6}{24}(6) \right]$$

$$= -0.25 + \frac{(2x-1)}{2}(2.5) + \frac{3x^2-6x+2}{6}(9) + \frac{4x^3-18x^2+22x-6}{24}(6)$$

$$= -0.25 + 2.5x - 1.25 + 4.5x^2 - 9x + 3 + x^3 - 4.5x^2 + 5.5x - 1.5$$

$$\frac{dy}{dx} = x^3 - x$$

Now $\frac{dy}{dx} = 0 \Rightarrow x^3 - x = 0$

$x(x^2-1) = 0$

$x = 0, \quad x^2 - 1 = 0$

$x = 0, \quad (x-1)(x+1) = 0$

$x = 0, \quad x = 1, \quad x = -1$

$$\frac{d^2y}{dx^2} = 3x^2 - 1$$

at $x = 0$, $\frac{d^2y}{dx^2} = -1 = -ve$

$x = 1$ $\quad \frac{d^2y}{dx^2} = 3 - 1 = 2 = +ve$

$x = -1$ $\quad \frac{d^2y}{dx^2} = 3 - 1 = 2 = +ve$

$\therefore$ y is maximum at $x = 0$ minimum at $x = 1$ and -1

$$\therefore y(x) = \frac{1}{h}\left[y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \cdots\right]$$

$$y(0) = \frac{1}{1}[0 + 0] = 0$$

$\therefore$ maximum value $= 0$

$$y(1) = \frac{1}{1}[y_0 + \Delta y_0 + 0 + 0 + \cdots]$$

$$= [0 + (-0.25)]$$

$$= -0.25$$

$\therefore$ maximum at $x = 1$, $y(1) = -0.25$.

④ consider the following table of data:

x	0.2	0.4	0.6	0.8	1.0
$f(x)$	0.9798652	0.9177710	0.8080348	0.6386093	0.3843735

Find $f'(0.25)$ using Newton's forward difference approximation $f'(0.6)$ using Stirling's approximation and $f'(0.95)$ using Newton's backward difference approximation.

Soln:

Here $h = 0.2$

Newton's forward interpolation formula for derivatives

$$y'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \dots\right]$$

where $u = \frac{x-x_0}{h}$ $x=0.25$, $x_0 = 0.2$, $h=0.2$ $u = \frac{0.25-0.2}{0.2} = 0.25$

The difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.2	0.9798652 (y_{-2})				
		-0.0620942 Δy_{-2}			
			-0.047642 $\Delta^2 y_{-2}$		
0.4	0.9177710 (y_{-1})			-0.0120473 $\Delta^3 y_{-2}$	
		-0.1097362 Δy_{-1}			-0.01310985 $\Delta^4 y_{-2}$
0.6	0.8080348 (y_0)		-0.0596893 $\Delta^2 y_{-1}$		
		-0.1694255 Δy_0		-0.02515715 $\Delta^3 y_{-1}$	
0.8	0.6386093 (y_1)		-0.08484645 $\Delta^2 y_0$		
		-0.25427195 Δy_1			
1.0	0.38433735 (y_2)				

$$y'(0.25) = \frac{1}{0.2}\Big[(-0.0620942) + \frac{(2(0.25)-1)}{2}(-0.047642)$$
$$+ \frac{3(0.25)^2-6(0.25)+2}{6}(-0.0120473)$$
$$+ \frac{4(0.25)^3-18(0.25)^2+22(0.25)-6}{24}(-0.01310985)\Big]$$

$$= \frac{1}{0.2}\left[-0.0620942 + 0.0119105 - 0.001380419 + 0.000853505\right]$$

$$= \frac{1}{0.2}\left[-0.050710613\right]$$

$$= -0.253553065$$

$= -0.2536$ [correct to four decimal places]

Stirling's formula for derivative is

$u = \frac{x-x_0}{h}$, $x=0.6$, $x_0 = 0.2$, $h=0.2$

$$u = \frac{0.6-0.2}{0.2} = \frac{0.4}{0.2} = 0.2$$

$$y'(x) = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} + u\,\Delta^2 y_{-1} + \frac{3u^2-1}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{4u^3-2u}{4!}\Delta^4 y_{-2} + \cdots\right]$$

$$y'(0.6) = \frac{1}{0.2}\left[\frac{-0.1694255 - 0.1097362}{2} + (0.2)\,(-0.00596893) + \frac{3(0.2)^2-1}{6}\left[\frac{-0.02515715 - 0.0120473}{2}\right] + \frac{4(0.2)^3 - 2(0.2)}{24}(-0.01310985)\right]$$

$$= \frac{1}{0.2}\left[-0.13958085 - 0.01193786 + (-0.14667)(-0.01860225) + (-0.01533)(-0.01310985)\right]$$

$$= \frac{1}{0.2}\left[-0.13958085 - 0.01193786 + 0.0027\,28388 + 0.000200974\right]$$

$$= \frac{1}{0.2}\left[-0.148589348\right]$$

$$= -0.74294674$$

$= -0.74295$ correct to five decimal places

Newton's backward difference formula

$$y'(x) = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2}\nabla^2 y_n + \frac{2u^2+6u+2}{6}\nabla^3 y_n + \frac{2u^3+9u^2+11u+3}{12}\nabla^4 y_n + \cdots\right]$$

$$u = \frac{x - x_n}{h} = \frac{0.95-1}{0.2} = -0.25$$

$$y'(0.95) = \frac{1}{h}\left[-0.25427195 + \frac{2(-0.95)+1}{2}(-0.08484645) + \frac{2(-0.25)^2 + 6(-0.25)+2}{6}(-0.02515715) + \frac{2(-0.25)^3 + 9(-0.25)^2 + 11(-0.25)+3}{12}(-0.01310985)\right]$$

$$= \frac{1}{0.2}\left[-0.25427195 + (0.25)(-0.08484645) + (0.104167)(-0.02515715) + (0.065104166)(-0.01310985)\right]$$

$$= \frac{1}{0.2}\left[-0.343208273\right]$$

$$= -1.71604$$

(5) Find the gradient of the road at the middle point of the elevation above a datum line of seven points of road which are given below:

x	0	300	600	900	1200	1500	1800
y	135	149	157	183	201	205	193

Solution:- We require $\left(\frac{dy}{dx}\right)_{x=900}$

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	135						
		14					
300	149						
			-6				
		8		24			
600	157				-56		
			-18				
		$\boxed{26}$		$\boxed{-26}$		70	
							-86
$\boxed{900}$	183 (y_0)		-8		20		
		$\boxed{18}$		$\boxed{-6}$		-16	
1200	201		-14		4		
		4		-2			
1500	205		-16				
		-12					
1800	193						

Since $x = 900$ is in the middle of the table we use Stirling's formula.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \dots\right]$$

$$= \frac{1}{300}\left[\frac{1}{2}(18+26) - \frac{1}{12}(-6-26) + \frac{1}{60}[70-16]\right]$$

$$= \frac{1}{300}\left[22 + 2.6666 + 0.9\right] = 0.085222$$

Hence the gradient of the road at the middle point is 0.084776.

⑥ Obtain the value of $f'(0.04)$ using Bessel's formula given in the table below:

x	0.01	0.02	0.03	0.04	0.05	0.06
$f(x)$	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

Soln:
Since $x = 0.04$ is in the middle of the table we use central difference formula and in particular Bessel's formula.

The central difference table is

x	u	y	Δy	$\Delta^2 y$	$\Delta^3 y$	Δ^4	$\Delta^5 y$
0.01	-3	0.1023					
			0.0024				
0.02	-2	0.1047		0.0			
			0.0024		0.0001		
0.03	-1	0.1071		0.0001		-0.0001	
			0.0025		0.0		0.0
0.04	0	0.1096 (y_0)		0.0001		-0.0001	
			0.0026		-0.0001		
0.05	1	0.1122		0.0 $(\Delta^2 y_0)$			
			0.0026				
0.06	2	0.1148					

Since $u = \dfrac{x - x_0}{h} = \dfrac{x - 0.04}{0.01}$

Taking $x_0 = 0.04$ as the origin

$y_0 = 0.1096 \quad \Delta y_0 = 0.0026 \quad \Delta y_{-1} = 0.0025, \quad \Delta y_{-2} = 0.0024$

By Bessel's formula

$$y(x_0 + uh) = \frac{1}{2}(y_0 + y_1)(u - \tfrac{1}{2})\Delta y_0 + \frac{u(u-1)}{4}(\Delta^2 y_{-1} + \Delta^2 y_0)$$
$$+ \frac{u(u - \tfrac{1}{2}(u-1))}{6}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{48}(\Delta^4 y_{-2} + \Delta^4 y_{-1})$$

$$y'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{4}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{(3u^2 - 3u + \tfrac{1}{2})}{6}\Delta^3 y_{-1} + \cdots\right]$$

$$y'(x_0) = \frac{1}{0.01}\left[0.0026 - \frac{1}{4}(0 + 0.0001) + \frac{1}{12}(-0.0001) + \frac{1}{24}(-0.0001)\right]$$

$$= \frac{1}{0.24}\left[24 \times 0.0026 - 0.0006 - 0.0003\right]$$

$$f'(0.04) = 0.25625.$$

Numerical integration by trapezoidal and Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rules, Romberg's method

Trapezoidal rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$$

This is known as the trapezoidal rule.

Simpson's one third rule.

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+\cdots+y_{n-1})+2(y_2+y_4+\cdots+y_{n-2})\right]$$

This is known as the Simpson's one-third rule (or) simply Simpson's rule and is most commonly used.

Note:-

While applying (3), the given interval must be divided into even number of equal sub-intervals, since we find the area of two strips at a time.

Simpson's three-eight rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-1}) + 2(y_3+y_6+\cdots+y_{n-3})\right]$$

which is known as Simpson's three-eight rule.

Note:- While applying (4) the number of sub-intervals should be taken multiple of 3.

Formula.

AT A GLANCE				
Rule	Degree of $y(x)$	No. of intervals	Error	order
Trapezoidal rule	one	any	$\lvert E\rvert < \frac{(b-a)h^2}{12}M$	h^2
Simpson's 1/3 rule	Two	even	$\lvert E\rvert < \frac{(b-a)h^4}{180}M$	h^4
Simpson's 3/8 rule	Three	multiple of 3	$\lvert E\rvert = \frac{3}{8}h^5$	

Problems:-

1. using trapezoidal rule evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ taking 8 intervals.

Soln:-

Here $y(x) = \frac{1}{1+x^2}$

Length of the interval $= 2$

So we divide 8 equal intervals with $h = \frac{2}{8} = 0.25$

we form a table

x :	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
y :	0.5	0.64	0.8	0.9412	1	0.9412	0.8	0.64	0.5

Trapezoidal rule,

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = \frac{h}{2}\left[\text{Sum of the first and last ordinates} + 2(\text{Sum of the remaining ordinates})\right]$$

$$= \frac{0.25}{2}\left[(0.5+0.5) + 2(0.64+0.8+0.9412+0.9412+0.8+0.64)\right]$$

$$= \frac{0.25}{2}\left[1 + 2(5.7624)\right]$$

$$= \frac{0.25}{2}\left[12.5248\right]$$

$$= 1.5656$$

(2) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ with $h=\frac{1}{6}$ by trapezoidal rule

Soln:-

Here $y(x)=\frac{1}{1+x^2}$, $h=1/6$

x	0	$\frac{1}{6}$	$2/6$	$3/6$	$4/6$	$5/6$	1
y	1	$\frac{36}{37}$	$9/10$	$4/5$	$9/13$	$\frac{36}{61}$	$1/2$

By Trapezoidal rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\Big[(\text{sum of the first and last oridinate})+2(\text{sum of the remaining ordinates})\Big]$$

$$= \frac{(1/6)}{2}\left[\left(1+1/2\right)+2\left(\frac{36}{37}+9/10+4/5+9/13+\frac{36}{61}\right)\right]$$

$$= \frac{1}{12}\left[3/2+2[3.9554]\right]$$

$$= \frac{1}{12}\left[3/2+7.9108\right]$$

$$= 0.7842.$$

(3) Evaluate the integral $\int_1^2 \frac{dx}{1+x^3}$ using Trapezoidal rule with two sub intervals.

Soln:-

Here $y(x)=\frac{1}{1+x^2}$, $h=1/2=0.5$

x	1	1.5	2
y	0.5	0.3077	0.2

By Trapezoidal rule

$$\int_1^2 \frac{dx}{1+x^3} = \frac{h}{2}\Big[\text{sum of the first and last oridinates}+2(\text{sum of the remaining ordinates})\Big]$$

$$= \frac{0.5}{2}\left[0.5+0.2+2(0.3077)\right]$$

$$= \frac{0.5}{2}\left[0.7+0.6154\right]$$

$$= \frac{0.5}{2}\left[1.3154\right] = 0.3289$$

④ dividing the rang into 10 equal parts, find the value of

$\int_0^{\pi/2} \sin x \, dx$ by (i) Trapezoidal rule

(ii) Simpson's rule

Soln:

x	0	$\frac{\pi}{20}$	$\frac{2\pi}{20}$	$\frac{3\pi}{20}$	$\frac{4\pi}{20}$	$\frac{5\pi}{20}$	$\frac{6\pi}{20}$	$\frac{7\pi}{20}$	$\frac{8\pi}{20}$	$\frac{9\pi}{20}$	$\frac{10\pi}{20}$
$y=\sin x$	0	0.1564	0.3090	0.4540	0.5878	0.7071	0.8090	0.8910	0.9511	0.9877	1

(i) By Trapezoidal rule

$$\int_0^{\pi/2} \sin x \, dx = \frac{h}{2}\left[y_0 + y_{11} + 2(y_1 + y_2 + \cdots + y_{10})\right]$$

$$h = \frac{\pi/2}{10} = \pi/20$$

$$\int_0^{\pi/2} \sin x \, dx = \frac{\pi}{40}(12.7062) = 0.9980$$

(ii) By Simpson's ⅓ rule

$$\int_0^{\pi/2} \sin x \, dx = (h/3)\left[(y_0 + y_{11}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8 + y_{10})\right]$$

$$= \left(\frac{\pi/20}{3}\right)\left[(0+1) + 4(3.1962) + 2(2.6569)\right]$$

$$= \frac{\pi}{60}\left[1 + 12.7848 + 5.3138\right]$$

$$= \frac{\pi}{60}\left[19.0986\right]$$

$$= 1.0000$$

⑤ using Simpson's one third rule evaluate $\int_0^1 x e^x \, dx$ taking 4 intervals. Compare your result with acutual value.

Soln:

x	0	0.25	0.5	0.75	1
$y = xe^x$	0	0.321	0.824	1.588	2.718

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\Big[\text{Sum of the first and last even ordinates} + 2(\text{Sum of remaining even ordinates}) + 4(\text{Sum of odd ordinates})\Big]$$

$$= \frac{0.25}{3}\left[(0+2.718) + 2(0.321+1.588) + 4(0.824)\right]$$

$$= \frac{0.25}{3}\left[2.718 + 3.818 + 3.296\right]$$

$$= \frac{2.458}{3} = 0.819 = 1$$

$$\int_0^1 xe^x\,dx = \int_0^1 x\,d(e^x) = \left[xe^x\right]_0^1 - \int_0^1 e^x\,dx$$

$$= (e^1 - 0) - (e^x)_0^1$$

$$= e - [e-1] = 1$$

(6) Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by (i) Trapezoidal rule (ii) Simpson's rule also check up the results by actual integration.

Soln:-

Here $b-a = 6-0 = 6$. Divide into 6 equal parts

$h = 6/6 = 1$. Hence, the table is

x	0	1	2	3	4	5	6
$\frac{1}{1+x^2} = f(x)$	1.00	0.500	0.200	0.100	0.058824	0.038462	0.27027

There are 7 ordinates $(n=6)$ we can use all the formula.

(i) By Trapezoidal rule,

$$I = \int_0^6 \frac{dx}{1+x^2} = \frac{1}{2}\left[(1+0.027027) + 2(0.5+0.2+0.1+0.058824+0.038462)\right]$$

$$= 1.41079950$$

(ii) By Simpson's one-third rule.

$$I = \frac{1}{3}\left[(1+0.027027) + 2(0.2+0.058824) + 4(0.5+0.1+0.038462)\right]$$

$= \frac{1}{3}(1.027027 + 0.517648 + 2.553848)$

$= 1.36617433$

(iii) By Simpson's $3/8$ rule.

$$I = \frac{3\times 1}{8}\left[(1+0.027027)+3(0.5+0.2+0.058824+0.038462)+2(0.1)\right]$$

$= 1.35708188$

(iv) By actual integration

$$I = \int_0^6 \frac{dx}{1+x^2} = \left(\tan^{-1}x\right)_0^6 = \tan^{-1}6 = 1.40564765$$

Conclusion:- Here the value by trapezoidal rule is closer to the actual value than the value by simpson's rule.

Romberg's method

① Evaluate $\int_0^2 \frac{dx}{x^2+4}$ using Romberg's method. Hence obtain an approximate value for π.

Solution:-

Let $y = \frac{1}{x^2+4}$ and Let $I = \int_0^2 \frac{dx}{x^2+4}$

Take $h=1$

The tabulated value of y are

x	0	1	2
y	0.25	0.20	0.125

using trapezoidal rule,

$$I_1 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}\left[(y_0+y_2)+2(y_1)\right]$$

$= (0.5)\left[(0.25+0.125)+2(0.20)\right]$

$= 0.3875$

Take $h=0.5$ The tabulated values of y are

x	0	0.5	1.0	1.5	2.0
y	0.25	0.2353	0.20	0.160	0.125

using Trapezoidal rule

$$I_2 = \frac{h}{2}\left[(y_0+y_4) + 2(y_1+y_2+y_3)\right]$$

$$=(0.25)\left[(0.25+0.125) + 2(0.2353+0.2+0.16)\right]$$

$$= 0.3914$$

Take $h=0.25$ The tabulated values of y are

x	0	0.50	0.75	1.0	1.25	1.50	1.75	2.00
y	0.25	0.2353	0.2192	0.20	0.1798	0.160	0.1416	0.125

By trapezoidal rule

$$I_3 = \frac{h}{2}\left[(y_0+y_8) + 2(y_1+y_2+\cdots+y_7)\right]$$

$$= \left(\frac{0.25}{2}\right)\left[(0.25+0.125) + 2(0.2462+0.2353+0.2192+0.20 +0.1798+0.16+0.1416)\right]$$

$$=(0.125)\left[3.1392\right]$$

$$I_3 = 0.3924$$

using Romberg's formula for I_1 and I_2 we have

$$I = I_2 + \left(\frac{I_2-I_1}{3}\right)$$

$$= 0.3914 + \left(\frac{0.3914-0.3875}{3}\right)$$

$$I = 0.3953 \quad \text{—— (1)}$$

using Romberg's formula for I_2 and I_3 we have

$$I = I_3 + \left(\frac{I_3-I_2}{3}\right)$$

$$= 0.3924 + \left(\frac{0.3924-0.3914}{3}\right)$$

$$= 0.3927 \quad \text{—— (2)}$$

Since (1) and (2) are not equal we go for one more application of Trapezoidal rule taking $h = 0.125$.

Take $h = 0.125$ The tabulated values are

x	0	0.125	0.250	0.375	0.500	0.625	0.750	0.875
y	0.25	0.249	0.2462	0.2415	0.2353	0.2278	0.2192	0.2098

1.00	1.125	1.250	1.375	1.500	1.625	1.750	1.875	2.000
0.20	0.1899	0.1798	0.1698	0.160	0.1506	0.1416	0.1331	0.125

By Trapezoidal rule

$$I_4 = \frac{h}{2}\left[(y_0 + y_{16}) + 2(y_1 + y_2 + \cdots + y_{15})\right]$$

$$= \left(\frac{0.125}{2}\right)\left[(0.25 + 0.125) + 2(0.249 + 0.2462 + \cdots + 0.1331)\right]$$

$$I_4 = 0.3926$$

Using Romberg's formula for I_3 and I_4 we have

$$I = I_4 + \left(\frac{I_4 - I_3}{3}\right)$$

$$= 0.3926 + \left(\frac{0.3926 - 0.3924}{3}\right)$$

$$I = 0.3927 \quad \text{—— (3)}$$

Since (2) and (3) are almost equal we can take

$$I = \int_0^2 \frac{dx}{x^2+4} = 0.3927 \quad \text{—— (4)}$$

By actual integration

$$\int_0^2 \frac{dx}{x^2+4} = \int_0^2 \frac{dx}{x^2+2^2} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big]_0^2$$

$$= \frac{1}{2}\left[\frac{\pi}{4}\right] = \frac{\pi}{8} \quad \text{—— (5)}$$

$\therefore$ From (4) and (5) we get $\pi/8 = 0.3927$

$\therefore \pi \approx 3.1416.$

(2) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Romberg's method correct to 4 decimal places. Hence deduce an approximate value of π.

Solution:

Let $y = \frac{1}{1+x^2}$ and let $I = \int_0^1 \frac{dx}{1+x^2}$

Take $h=0.5$ The tabulated values of y are

x	0	0.5	1
$y=\frac{1}{1+x^2}$	1	0.8	0.5

using Trapezoidal rule

$$I_1 = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_2)+2y_1\right]$$

$$= \frac{0.5}{2}\left[(1+0.5)+1.6\right]$$

$$= 0.775$$

Take $h=0.25$ The tabulated values of y are

x	0	0.25	0.50	0.75	1.00
$y=\frac{1}{1+x^2}$	1	0.9412	0.80	0.64	0.5

using Trapezoidal rule,

$$I_2 = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$$

$$= \frac{0.25}{2}\left[(1+0.5)+2(0.9412+0.80+0.64)\right]$$

$$= 0.7828$$

Take $h=0.125$ The tabulated values of y are

x	0	0.125	0.25	0.375	0.50	0.625	0.750	0.875	1.0
$y=\frac{1}{1+x^2}$	1	0.9846	0.9412	0.8767	0.80	0.7191	0.64	0.5664	0.5

using Trapezoidal rule

$$I_3 = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_8)+2(y_1+y_2+\cdots+y_7)\right]$$

$$= \left(\frac{0.125}{2}\right)\left[(1+0.5)+2(0.9846+0.9412+0.8767+0.8\right.$$
$$\left.+0.7191+0.64+0.5664)\right]$$

$= (0.0625)[1.5 + 2(5.528)]$

$= 0.78475$

using Romberg's formula for I_1 and I_2 we have

$$I = I_2 + \left(\frac{I_2 - I_1}{3}\right)$$

$$= 0.7828 + \left(\frac{0.7828 - 0.775}{3}\right)$$

$$= 0.7828 + 0.0026$$

$$= 0.7854$$

using Romberg's formula for I_2 and I_3 we have

$$I = I_3 + \left(\frac{I_3 - I_2}{3}\right) = 0.78475 + \left(\frac{0.78475 - 0.7828}{3}\right)$$

$$= 0.78475 + 0.00065$$

$$= 0.7854$$

$$\therefore I = \int_0^1 \frac{dx}{1+x^2} = 0.7854 \quad \text{--- (1)}$$

By actual evaluation of the definite integral we have

$$I = \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \pi/4 \quad \text{--- (2)}$$

From (1) and (2) we have $\pi/4 = 0.7854$.

Hence $\pi \approx 3.1416$.

TWO AND THREE POINT GAUSSIAN QUADRATURE FORMULAS

Two points Gaussian quadrature - Problems

Formula:-

$$\int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

This formula is exact for polynomials upto degree 3.

① Apply Gauss two point formula to evaluate $\int_{-1}^{1} \frac{1}{1+x^2} dx$

Soln:

Given interval is -1 to 1 so we apply

$$\int_{-1}^{1} f(x)\, dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \text{ formula.}$$

Here $f(x) = \frac{1}{1+x^2}$

$$f\left(\frac{-1}{\sqrt{3}}\right) = \frac{1}{1+\frac{1}{3}} = \frac{1}{\left(\frac{4}{3}\right)} = \frac{3}{4}$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{1+\frac{1}{3}} = \frac{1}{\left(\frac{4}{3}\right)} = \frac{3}{4}$$

$$\therefore \int_{-1}^{1} f(x)\, dx = \int_{-1}^{1} \frac{1}{1+x^2} dx = \frac{3}{4} + \frac{3}{4} = \frac{3}{2} = 1.5$$

But actual integration

$$\int_{-1}^{1} \frac{1}{1+x^2} dx = \left[\tan^{-1} x\right]_{-1}^{1} = \tan^{-1}(1) - \tan^{-1}(-1)$$

$$= \tan^{-1}(1) + \tan^{-1}(1)$$

$$= 2\tan^{-1}(1)$$

$$= 2\, \frac{\pi}{4}$$

$$= \frac{\pi}{2} = 1.5708$$

Here the error due to two-point formula is 0.0708

②. Apply Gauss two-point formula to evaluate $\int_{0}^{1} \frac{dx}{1+x^2}$

Soln:

Given interval is 0 to 1, to make them as -1 to 1

$$\int_{0}^{1} \frac{dx}{1+x^2} = \frac{1}{2}\int_{-1}^{1} \frac{dx}{1+x^2} \quad \left[\because \frac{1}{1+x^2} \text{ is an even function}\right]$$

$$= \frac{1}{2}[1.5] \quad [\text{by first problem}]$$

$$= 0.75$$

③. using Gaussian two-point formula evaluate

(i) $\int_{-1}^{1} (3x^2 + 5x^4)\, dx$ (ii) $\int_{0}^{1} (3x^2+5x^4)\, dx$

Solution:-

(i) given interval is -1 to 1

Hence we can apply the formula

$$\int_{-1}^{1} f(x)\, dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Here $f(x) = 3x^2 + 5x^4$

$$f\left(-1/\sqrt{3}\right) = 3\left(1/3\right) + 5\left(1/9\right) = 1 + 5/9 = \frac{14}{9} = 1.556$$

$$f\left(+1/\sqrt{3}\right) = 3\left(1/3\right) + 5\left(1/9\right) = 1 + 5/9 = \frac{14}{9} = 1.556$$

$$\int_{-1}^{1} (3x^2+5x^4)\, dx = (1.556 + 1.556) = 3.112$$

(ii) Given interval is 0 to 1, so to make them as -1 to 1

Soln:- $$\int_{0}^{1} (3x^2+5x^4)\, dx = \frac{1}{2}\int_{-1}^{1} (3x^2+5x^4)\, dx$$

[$\because 3x^2+5x^4$ is an even function]

$$= \frac{1}{2}[3.112] = 1.556$$

④ Evaluate $\int_{-2}^{2} e^{-x/2}\, dx$ by Gauss two point formula.

Soln:-

Given The range is not $(-1, 1)$ so by using the formula to make them as $(-1, 1)$

$$x = \frac{b-a}{2} z + \frac{b+a}{2}.$$ Here $a = -2$; $b = 2$

$$x = \frac{2+2}{2} z + \frac{2-2}{2}$$

$$x = 2z \Rightarrow z = x/2$$

$$dx = 2dz$$

$$\int_{-2}^{2} e^{-x/2}\,dx = \int_{-1}^{1} e^{-z}(2dz)$$

$$= 2\int_{-1}^{1} e^{-z}\,dz$$

$$= 2\left[f(-1/\sqrt{3}) + f(1/\sqrt{3})\right]$$

Here $f(z) = e^{-z}$

$$f(-1/\sqrt{3}) = e^{1/\sqrt{3}} = 1.7813$$

$$f(1/\sqrt{3}) = e^{-1/\sqrt{3}} = 0.5614$$

$$= 2\,[0.5614 + 1.7813]$$

$$= 4.6854$$

Three points Gaussian quadrature

Formula $\int_{-1}^{1} f(x)\,dx = \frac{5}{9}\left[f(-\sqrt{3/5}) + f(\sqrt{3/5})\right] + \frac{8}{9}f(0)$

This formula is exact for polynomials upto degree 5.

① using Gaussian three - point formula evaluate

(i) $\int_{-1}^{1} (3x^2 + 5x^4)\,dx$

(ii) $\int_{0}^{1} (3x^2 + 5x^4)\,dx$

also compare with exact values

Soln:-

Let $f(x) = 3x^2 + 5x^4$ [Range given is exact form]

$f(0) = 0$

$$f(-\sqrt{3/5}) = 3(3/5) + 5(3/5)^2 = 9/5 + 9/5 = 18/5$$

$$f(\sqrt{3/5}) = 3(3/5) + 5(3/5)^2 = 9/5 + 9/5 = 18/5$$

$$\therefore \int_{-1}^{1} f(x)\,dx = \frac{5}{9}\left[f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right)\right] + \frac{8}{9} f(0)$$

$$= \frac{5}{9}\left[\frac{18}{5} + \frac{18}{5}\right] + 0$$

$$= \frac{5}{9}\,\frac{36}{5} = 4 \quad -①$$

Exact value

$$\int_{-1}^{1} (3x^2+5x^4)\,dx = 2\int_{0}^{1} (3x^2+5x^4)\,dx \quad [\because 3x^2+5x^4 \text{ is an even function}]$$

$$= 2\left[\frac{3x^3}{3} + \frac{5x^5}{5}\right]_0^1$$

$$= 2\left[x^3 + x^5\right]_0^1$$

$$= 2\left[(1+1) - (0+0)\right]$$

$$= 4$$

we get exact value by using Gaussian three-point formula

(ii) $\int_0^1 (3x^2+5x^4)\,dx$ [The range is not exact form]

$$\int_0^1 (3x^2+5x^4)\,dx = \frac{1}{2}\int_{-1}^{1} (3x^2+5x^4)\,dx \quad [\because 3x^2+5x^4 \text{ is an even function}]$$

$$= \frac{1}{2}[4] = 2 \quad [\text{by (1)}]$$

② using three-point Gaussian quadrature formula, evalute

(i) $\int_{-1}^{1} \frac{1}{1+x^2}\,dx$

(ii) $\int_0^1 \frac{1}{1+t^2}\,dt$

Soln:

Let $f(x) = \frac{1}{1+x^2}$ [Range given is exact form]

$$f(0) = \frac{1}{1+0} = 1$$

$$f\left(-\sqrt{3/5}\right) = \frac{1}{1+3/5} = \frac{1}{8/5} = 5/8$$

$$f\left(-\sqrt{3/5}\right) = \frac{1}{1+3/5} = \frac{1}{8/5} = 5/8$$

Three-point Gaussian quadrature formula is

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = 5/9\left[f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right)\right] + 8/9\, f(0)$$

$$= 5/9\left[5/8 + 5/8\right] + 8/9\,(1)$$

$$= 5/9\left[2(5/8)\right] + 8/9$$

$$= \frac{50}{72} + 8/9 = \frac{19}{12} = 1.5833 \quad \text{--- (1)}$$

Actual value

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = 2\int_{0}^{1} \frac{1}{1+x^2}\,dx \quad \left[\frac{1}{1+x^2} \text{ is an even formula}\right]$$

$$= 2\left[\tan^{-1} x\right]_0^1$$

$$= 2\left[\tan^{-1} 1 - \tan^{-1} 0\right]$$

$$= 2\left[\pi/4\right]$$

$$= \pi/2 = 1.5708$$

(ii) Range given is not exact form

$$\therefore \int_{0}^{1} \frac{1}{1+t^2}\,dt = 1/2\int_{-1}^{1} \frac{1}{1+t^2}\,dt \quad \left[\because \frac{1}{1+t^2} \text{ is an even function}\right]$$

$$= 1/2\left[1.5833\right] = 0.79165$$

(3) Evaluate $\int_0^2 \frac{x^2+2x+1}{1+(x+1)^4}\,dx$ by Gaussian three point formula

Soln:-

Let $f(x) = \frac{x^2+2x+1}{1+(x+1)^4}$ [Range given is not ~~the~~ exact form]

Let $x = \frac{b-a}{2} z + \frac{b+a}{2}$ $\quad [a=0, b=2]$

$= \frac{2-0}{2} z + \frac{2+0}{2}$

$x = z+1$ | $x = 0 \Rightarrow z = -1$

$dx = dz$ | $x = 2 \Rightarrow z = 1$

$$\int_0^2 \frac{x^2+2x+1}{1+(x+1)^4} dx = \int_{-1}^{1} \frac{(z+1)^2 + 2(z+1)+1}{1+[(z+1)+1]^4} dz$$

$$= \int_{-1}^{1} \frac{z^2+2z+1+2z+2+1}{1+(z+2)^4} dz$$

$$= \int_{-1}^{1} \frac{z^2+4z+4}{(z+2)^4+1} dz \quad \text{—} ①$$

[Range given is in exact form]

$$\therefore f(z) = \frac{z^2+4z+4}{(z+2)^4+1}$$

$$f(z) = \frac{(z+2)^2}{(z+2)^4+1}$$

$$f(0) = \frac{2^2}{2^4+1} = \frac{4}{17}$$

$$f\left[-\sqrt{3/5}\right] = \frac{\left[-\sqrt{3/5}+2\right]^2}{\left[-\sqrt{3/5}+2\right]^4+1} = \frac{1.50161}{3.2548} = 0.4614$$

$$f\left[+\sqrt{3/5}\right] = \frac{\left[\sqrt{3/5}+2\right]^2}{\left[\sqrt{3/5}+2\right]^4+1} = \frac{7.69839}{60.2652} = 0.12774$$

$$\therefore (1) \Rightarrow \int_{-1}^{1} f(z)\,dz = 5/9\left[f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right)\right] + 8/9\, f(0)$$

$$= 5/9\left[0.4614 + 0.12774\right] + 8/9\left[\frac{4}{17}\right]$$

$$= 0.3273 + 0.2092 = 0.5365$$

$$\therefore (1) \Rightarrow \int_0^2 \frac{x^2+2x+1}{1+(x+1)^4} dx = \int_{-1}^{1} \frac{z^2+4z+4}{(z+2)^2+1} dz = 0.5365$$

DOUBLE INTEGRALS USING TRAPEZOIDAL AND SIMPSON'S RULES

Trapezoidal rule

$$I = \frac{hk}{4}\left[\text{sum of the values of } f(x,y) \text{ at the four corner points}\right]$$

Simpson's rule for double integration

$I = \frac{hk}{9}\Big[$ (sum of the values of f at the four corners)

+ 2 (sum of the values of f at the odd positions on the boundary except the corners)

+ 4 (sum of the values of f at the even positions on the boundary)

+ {4 (sum of the values of f at odd poistions)

8 (sum of the values of f at even poistions) on the odd row f of the matrix except boundary rows}

+ 8{ (sum of the values of f at the odd poisitions)

+16 (sum of the values of f at the even poisitions) on the even rows of the matrix)}$\Big]$

problems:-

① Evaluate $\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\,dx\,dy$ using Trapezoidal and Simpson's rule verify your result by actual integration.

Solution:-

Divide the range of x and y into 4 equal parts.

$$h = \frac{2.4-2}{4} = 0.1 \quad \text{and} \quad k = \frac{1.4-1}{4} = 0.1$$

Get the values of $f(x,y) = \frac{1}{xy}$ at nodal points.

y/x	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3968	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

Case (i): By Trapezoidal rule, we get

$I = \frac{hk}{4}$ [(sum of values of f at the four corners

+ 2 (sum of values of f at the remaining nodes on the boundary) + 4 (sum of the values of f at the interior nodes)]

$$= \frac{(0.1)(0.1)}{4}\left[(0.5) + 0.4167 + 0.3571 + 0.2976\right]$$

$$+2\left[0.3846 + 0.4167 + 0.4545 + 0.4762 + 0.4545 + 0.4348 + 0.3788 + 0.3472 + 0.3205 + 0.3106 + 0.3247 + 0.3401\right]$$

$$+4\left[0.4329 + 0.4132 + 0.3953 + 0.3968 + 0.3788 + 0.3623 + 0.3663 + 0.3497 + 0.3344\right]$$

$$= \frac{0.01}{4}\left[1.5714 + 9.2864 + 13.7188\right]$$

$$= 0.0614$$

Case (ii): By Simpson's rule

$I = \frac{hk}{9}$ [(sum of the values of f at the four corners)

+2(sum of the values of f at the odd positions on the boundary except the corners)

+4 (sum of the values of f at the even positions on the boundary)

+ {4 (sum of the values of f at odd positions)

+ 8 (sum of the values of f at even positions) on the odd row of the matrix except boundary rows}

+{8 (sum of values of f at the odd positions)

+16 (sum of values of f at the even positions) on the even rows of the matrix}]

$$= \frac{(0.1)(0.1)}{9}\Big[(0.5 + 0.4167 + 0.3571 + 0.2976)$$

$$+ 2\,(0.4167 + 0.4545 + 0.3472 + 0.3247)$$

$$+ 4\,(0.3846 + 0.4545 + 0.4762 + 0.4348 + 0.3788 + 0.3205 + 0.3106 + 0.3401)$$

$$+ 4\,(0.3788)$$

$$+ 8\,(0.3968 + 0.3623)$$

$$+ 8\,(0.3497 + 0.4132)$$

$$+ 16\,(0.3663 + 0.3344 + 0.4329 + 0.3953)\Big]$$

$$= \frac{0.01}{9}\,[55.2116] = 0.0657$$

Case (3) :- By actual integration

$$\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\,dx\,dy = \left(\int_1^{1.4} \frac{1}{y}\,dy\right)\left(\int_2^{2.4} \frac{1}{x}\,dx\right)$$

$$= (\log y)_1^{1.4}\,(\log x)_2^{2.4}$$

$$= (\log 1.4)\,[\log 2.4 - \log 2]$$

$$= \log(1.4)\,\log(1.2)$$

$$= 0.0613.$$

we get the actual value and the value by simpson's rule are equal while the value by trapezoidal rule difers only by 0.0001.

② Evaluate $\int_0^2\int_0^2 f(x,y)\,dx\,dy$ by Trapezoidal rule for the following data.

y/x	0	0.5	1	1.5	2
0	2	3	4	5	5
1	3	4	6	9	11
2	4	6	8	11	14

Solution:-

Here $h = 0.5$

$k = 1$

$$I = \int_0^2 \int_0^2 f(x, y)\, dx\, dy$$

$I = \frac{hk}{4}\Big[$(sum of values of f at the four corners)

+ 2 (sum of the values of f at the remaining nodes on the boundary)

+ 4 (sum of the values of f at the interior nodes)$\Big]$

$$= \frac{(0.5)(1)}{4}\left[(2+5+14+4) + 2(3+3+4+5+11+11+8+6) + 4(4+6+9)\right]$$

$$= \frac{(0.5)(1)}{4}\left[25 + 2(51) + 4(19)\right]$$

$$= (0.125)[203]$$

$$= 25.375$$

③ Using Simpson's $\frac{1}{3}$ rule evaluate $\int_0^1 \int_0^1 \frac{1}{1+x+y}\, dx\, dy$ taking $h = k = 0.5$

Soln:

y/x	0	0.5	1
0	1	0.6667	0.5
0.5	0.6667	0.5	0.4
1	0.5	0.4	0.3333

Simpson's rule:-

$I = \frac{hk}{9}\Big[$(sum of the values of f at the four corners)

+ 2 (sum of the values of f at the odd positions on the boundary except the corners)

+ 4 (sum of the values of f at the even positions on the boundary)

+ {4 (sum of the values of f at odd positions)

+ 8 (sum of the values of f at even positions) on the odd row of the matrix except boundary rows}

$+\{8$ (sum of the values of f at the odd positions)

$+16$ (sum of the values of f at the even positions) on the even rows of the matrix$\}]$

$$I = \frac{(0.5)(0.5)}{9}\left[(1+0.5+0.3333+0.5) + 2(0) + 4(0.6667+0.6667+0.4+0.4) + \{4(0)+8(0)\} + \{8(0)+16(0.5)\}\right]$$

$$= (0.02778)[(2.3333) + 4(2.1334) + 8]$$

$$= (0.02778)(18.8669)$$

$$= 05241$$

4) Evaluate $\int_1^2\int_1^2 \frac{dx\,dy}{x^2+y^2}$ numerically with $h=0.2$ along x-direction and $k=0.25$ along y-direction.

Solution:-

y/x	1	1.2	1.4	1.6	1.8	2
1	0.5	0.4098	0.3378	0.2809	0.2359	0.2
1.25	0.3902	0.3331	0.2839	0.2426	0.2082	0.1798
1.5	0.3077	0.2710	0.2375	0.2079	0.1821	0.16
1.75	0.2462	0.2221	0.1991	0.1779	0.1587	0.1416
2	0.2	0.1838	0.1679	0.1524	0.1381	0.125

By Trapezoidal rule

$$\int_1^2\int_1^2 \frac{1}{x^2+y^2}\,dx\,dy = \frac{hk}{4}\Big[\text{sum of values of } f \text{ at the four corners}$$

$+ 2$ (sum of the values of f at the remaining nodes on the boundary)

$+4$ (sum of the values of f at the interior nodes)$\Big]$

$$= \frac{(0.2)(0.25)}{4}\Big[(0.5+0.2+0.125+0.2) + 2(0.2462+0.3077+0.3902+0.4098+0.3378+0.2809+0.2359+0.1798+0.16+0.1416+0.1381+0.1524+0.1679+0.1838) + 4(0.3331+0.2839+0.2426+0.2082+0.2710+0.2375+0.2079+0.1821+0.2221+0.1991+0.1779+0.1587+0.1838+0.1679+0.1524+0.1381)\Big]$$

$$= (0.0125)\left[1.025 + 6.6566 + 13.4652\right]$$

$$= 0.2643$$