$$AX_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12\begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = 4.12\,X_5$$

$$AX_5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9706 \\ 1.9902 \\ 0 \end{bmatrix} = 3.9706\begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = 3.9706\,X_6$$

$$AX_6 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.0072 \\ 2.0024 \\ 0 \end{bmatrix} = 4.0072\begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = 4.0072\,X_7$$

$$AX_7 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9982 \\ 1.9994 \\ 0 \end{bmatrix} = 3.9982\begin{bmatrix} 1 \\ 0.5000 \\ 0 \end{bmatrix} = 3.9982\,X_8$$

$$AX_8 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = 4X_9$$

$\therefore$ Dominant eigen value $= 4$; corresponding eigen vector is $(1, 0.5, 0)$

To find the least eigen value, let $B = A - 4I$, since $\lambda_1 = 4$

$$\therefore B = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

we will find the dominant eigen value of B.

let $Y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial vector.

$$BY_1 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -3Y_2$$

$$BY_2 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.6666 \\ 0 \end{bmatrix} = -5\begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -5Y_3$$

$$\therefore BY_3 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.6666 \\ 0 \end{bmatrix} = -5\begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix}$$

$\therefore$ Dominant eigen value of B is -5

Adding 4, smallest eigen value of A $= -5+4 = -1$

Sum of eigen values = trace of A $= 1+2+3 = 6$

$4 + (-1) + \lambda_3 = 6, \quad \therefore \lambda_3 = 3$

All the three eigen values are, 4, 3, -1.

UNIT – II

INTERPOLATION AND APPROXIMATION.

LAGRANGIAN POLYNOMIALS

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3 + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n$$

①. Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(3)$ for

x	x_0 0	x_1 1	x_2 2	x_3 5
$f(x)$	2 y_0	3 y_1	12 y_2	147 y_3

Soln: By Lagrange's interpolation formula, we have

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3) + \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$= \frac{(x-1)(x-2)(x-5)}{(-10)}(2) + \frac{x(x-2)(x-5)}{4}(3) + \frac{(x-1)(x-5)}{-6}(12) + \frac{x(x-1)(x-2)}{60}(147)$$

$$y = f(3) = \frac{(3-1)(3-2)(3-5)}{-10}(2) + \frac{3(3-2)(3-5)}{4}(3) + \frac{3(3-1)(3-5)}{-6}(12) + \frac{3(3-1)(3-2)}{60}(147)$$

$$y = \frac{(2)(1)(-2)}{(-10)}(2) + \frac{(3)(1)(-2)}{4}(3) + \frac{3(2)(-2)}{(-6)}(12) + \frac{(3)(2)(1)}{60}(147)$$

$$= \frac{4}{10}(2) - \frac{6}{4}(3) + 2(12) + \frac{1}{10}(147)$$

$$= \frac{8}{10} - \frac{18}{4} + 24 + \frac{147}{10}$$

$$= 35 \text{ Ans}$$

④ Find the third degree polynomial $f(x)$ satisfying the following data.

x	1	3	5	7
y	24	120	336	720

Soln:

The Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-3)(x-5)(x-7)}{(1-3)(1-5)(1-7)}(24) + \frac{(x-1)(x-5)(x-7)}{(3-1)(3-5)(3-7)}(120) + \frac{(x-1)(x-3)(x-7)}{(5-1)(5-3)(5-7)}(336)$$
$$+ \frac{(x-1)(x-3)(x-5)}{(7-1)(7-3)(7-5)}(720)$$

$$= -\tfrac{1}{2}(x-3)(x-5)(x-7) + \frac{15}{2}(x-1)(x-5)(x-7) - 21(x-1)(x-3)(x-7)$$
$$+ 15(x-1)(x-3)(x-5)$$

$$= -\tfrac{1}{2}[x^3-15x^2+71x-105] + \frac{15}{2}[x^3-13x^2+47x-35] - 21[x^3-11x^2+31x-21]$$
$$= +15[x^3-9x^2+23x-15]$$

$$= \left[-\tfrac{1}{2} + \frac{15}{2} - 21 + 15\right]x^3 + \left[\frac{15}{2} - \frac{195}{2} + 231 - 135\right]x^2 + \left[\frac{-71}{2} + \frac{705}{2} - 605 + 345\right]x$$
$$+ \left[\frac{105}{2} - \frac{525}{2} + 441 - 225\right]$$

$$= x^3 + 6x^2 + 11x + 6$$

$$f(4) = 4^3 + 6(4^2) + 11(4) + 6$$
$$= 64 + 96 + 44 + 6$$
$$= 210$$

3. Using Lagrange's interpolation formula find $f(4)$ given that $f(0) = 2$, $f(1) = 3$, $f(2) = 12$, $f(15) = 3587$.

Soln:

Given

x	0	1	2	15
y	2	3	12	3587

By Lagrange's formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y = f(4) = \frac{(4-1)(4-2)(4-15)}{(0-1)(0-2)(0-15)}(2) + \frac{(4-0)(4-2)(4-15)}{(1-0)(1-2)(1-15)}(3)$$

$$+ \frac{(4-0)(4-1)(4-15)}{(2-0)(2-1)(2-15)}(12) + \frac{(4-0)(4-1)(4-2)}{(15-0)(15-1)(15-2)} 3587$$

$$= \frac{(3)(2)(-11)}{(-1)(-2)(-15)}(2) + \frac{(4)(2)(-11)}{(1)(-1)(-14)}(3) + \frac{(4)(3)(-11)}{(2)(1)(-13)}(12)$$

$$+ \frac{(4)(3)(2)}{(15)(14)(13)}(3587)$$

$$= \frac{132}{30} - \frac{264}{14} + \frac{1584}{26} + \frac{86088}{2730}$$

$$= 78$$

4. Find the missing term in the following table using Lagrange's interpolation.

x	0	1	2	3	4
y	1	3	9	–	81

Soln:

By Lagrange's interpolation formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Let $x_0 = 0 \quad y_0 = 1$

$x_1 = 1 \quad y_1 = 3$

$x_2 = 2 \quad y_2 = 9$

$x_3 = 4 \quad y_3 = 81$

$$y = f(x) = \frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)}(1) + \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)}(3) + \frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)}(9) + \frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)}(81)$$

$$f(3) = \frac{(3-1)(3-2)(3-4)}{(-1)(-2)(-4)}(1) + \frac{(3-0)(3-2)(3-4)}{(1)(-1)(-3)}(3) + \frac{(3-0)(3-1)(3-4)}{(2)(-2)(1)}(9) + \frac{(3-0)(3-1)(3-2)}{(4)(3)(2)}(81)$$

$$= \frac{(2)(1)(-1)}{(-1)(-2)(-4)}(1) + \frac{(3)(1)(-1)}{(1)(-1)(-3)}(3) + \frac{(3)(2)(-1)}{(2)(1)(-2)}(9) + \frac{(3)(2)(1)}{(4)(3)(2)}(81)$$

$$= \frac{-2}{-8}(1) - 3 + \frac{27}{2} + \frac{81}{4}$$

$$= \frac{1}{4} - 3 + \frac{27}{2} + \frac{81}{4}$$

$$= 31$$

⑤ Find the parabola of the form $y = ax^2 + bx + c$ passing through the points $(0,0)$ $(1,1)$ and $(2,20)$

Soln:

we use Lagrange's interpolation formula

$$y = f(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)} \cdot 0 + \frac{(x-0)(x-2)}{(1-0)(1-2)} \cdot 1 + \frac{(x-0)(x-1)}{(2-0)(2-1)} \cdot 20.$$

$$= 0 - x(x-2) + 10x(x-1)$$

$$y = 9x^2 - 8x$$

Inverse interpolation

Taking y as independent variable

$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)} x_0 + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)} x_1 + \cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-0)(y_n-y_1)\cdots(y_n-y_{n-1})} x_n$$

This is called formula of inversion interpolation.

① Find the age corresponding to the annuity value 13.6 given the table.

Age (x):	30	35	40	45	50
Annuity Value (y)	15.9	14.9	14.1	13.3	12.5

Soln:-

$$x = \frac{(13.6-14.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(15.9-14.9)(15.9-14.1)(15.9-13.3)(15.9-12.5)} \times 30$$
$$+ \frac{(13.6-15.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(14.9-15.9)(14.9-14.1)(14.9-13.3)(14.9-12.5)} \times 35$$
$$+ \frac{(13.6-15.9)(13.6-14.9)(13.6-13.3)(13.6-12.5)}{(14.1-15.9)(14.1-14.9)(14.1-13.3)(14.1-12.5)} \times 40$$
$$+ \frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-12.5)}{(13.3-15.9)(13.3-14.9)(13.3-14.1)(13.3-12.5)} \times 45$$
$$+ \frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-13.3)}{(12.5-15.9)(12.5-14.9)(12.5-14.1)(12.5-13.3)} \times 50$$

$\therefore x(y=13.6) = 43$

② Find the value of θ given $f(\theta) = 0.3887$ where $f(\theta) = \int_0^\theta \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$ using the table

θ	$21°$	$23°$	$25°$
$f(\theta)$	0.3706	0.4068	0.4433

Soln: Now take $f(\theta)$ as independent and θ as dependent

$y = f(\theta)$:	0.3706	0.4068	0.4433
θ :	21	23	25

$$\theta = \frac{(y-0.4068)(y-0.4433)}{(0.3706-0.4068)(0.3706-0.4433)} \times 21 + \frac{(y-0.3706)(y-0.4433)}{(0.4068-0.3706)(0.4068-0.4433)} (23)$$

$$+ \frac{(y-0.3706)(y-0.4068)}{(0.4433-0.3706)(0.4433-0.4068)} \times 25$$

$$\theta(y=0.3887) = \frac{(0.3887-0.4068)(0.3887-0.4433)}{(0.3706-0.4068)(0.3706-0.4433)} \times 21$$

$$+ \frac{(0.3887-0.3706)(0.3887-0.4433)}{(0.4068-0.3706)(0.4068-0.4433)} \times 23$$

$$+ \frac{(0.3887-0.3706)(0.3887-0.4068)}{(0.4433-0.3706)(0.4433-0.4068)} \times 25$$

$$= 7.885832 + 17.202739 - 3.086525$$

$$= 22.0020$$

DIVIDED DIFFERENCES

TABLE

Argument x	Entry $f(x)$	First divided difference $\Delta f(x)$	Second divided difference $\Delta_2 f(x)$	Third divided difference $\Delta_3 f(x)$
x_0	$f(x_0)$			
		$f(x_0, x_1)$		
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$	
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$	
		$f(x_2, x_3)$		$f(x_1, x_2, x_3, x_4)$
x_3	$f(x_3)$		$f(x_2, x_3, x_4)$	
		$f(x_3, x_4)$		
x_4	$f(x_4)$			

① Form the divided difference table for the following data:-

x	1	2	3	4	7	12
$f(x)$	22	30	82	82	106	206

Soln:
The divided difference table is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	22				
		$\frac{30-22}{2-1}=8$			
2	30		$\frac{26-8}{4-1}=6$		
		$\frac{82-30}{4-2}=26$		$\frac{-3.6-6}{7-1}=-1.6$	
4	82		$\frac{8-26}{7-2}=-3.6$		$\frac{0.51+1.6}{12-1}=0.19$
		$\frac{106-82}{7-4}=8$		$\frac{1.5+3.6}{12-2}=0.51$	
7	106		$\frac{20-8}{12-4}=1.5$		
		$\frac{206-106}{12-7}=20$			
12	206				

(3) Show that $\underset{bcd}{\Delta^3}\left(\frac{1}{a}\right) = -\frac{1}{abcd}$

Soln:

If $f(x) = \frac{1}{x}$, $f(a) = \frac{1}{a}$

$$f(a,b) = \underset{b}{\Delta}\left(\frac{1}{a}\right) = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\frac{1}{ab}$$

$$f(a,b,c) = \frac{f(b,c)-f(a,b)}{c-a} = \frac{-\frac{1}{bc}+\frac{1}{ab}}{c-a} = \frac{1}{abc}\left(\frac{c-a}{c-a}\right) = \frac{1}{abc}$$

$$f(a,b,c,d) = \frac{f(b,c,d)-f(a,b,c)}{d-a} = \frac{\frac{1}{bcd}-\frac{1}{abc}}{d-a} = \frac{1}{abcd}\left(\frac{a-d}{d-a}\right) = -\frac{1}{abcd}$$

$$\therefore \underset{bcd}{\Delta^3}\left(\frac{1}{a}\right) = -\frac{1}{abcd}$$

Newton's divided difference formula (or) Newton's interpolation for unequal intervals

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})f(x_0,x_1,\ldots,x_n)$$

① using Netwon's divided difference formula, find $u(3)$ given $u(1) = -26$, $u(2) = 12$, $u(4) = 256$, $u(6) = 844$.

Soln:

We form the divided difference table since the intervals are unequal.

x	$u(x)$	$\Delta u(x)$	$\Delta^2 u(x)$	$\Delta^3 u(x)$
1	-26			
		$\frac{12+26}{2-1} = 38$		
2	12		$\frac{122-38}{4-1} = 28$	
		$\frac{256-12}{4-2} = 122$		$\frac{43-28}{6-1} = 3$
4	256		$\frac{294-122}{6-2} = 43$	
		$\frac{844-256}{6-4} = 294$		
6	844			

By Newton's divided difference interpolation formula

$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + \cdots$

Here

$u(x) = u(x_0) + (x-x_0) u(x_0, x_1) + (x-x_0)(x-x_1) u(x_0, x_1, x_2) + \cdots$

$x_0 = 1, \quad x_1 = 2, \quad x_2 = 4, \quad x_3 = 6$

$u(x_0) = -26, \quad u(x_0, x_1) = 38 \quad u(x_0, x_1, x_2) = 28, \quad u(x_0, x_1, x_2, x_3) = 3$

$\therefore u(x) = -26 + (x-1) 38 + (x-1)(x-2) 28 + (x-1)(x-2)(x-4) 3$

$\therefore u(3) = -26 + (2)(38) + (2)(1)(28) + (2)(1)(-1)(3)$

$= -26 + 76 + 56 - 6$

$u(3) = 100.$

② Find $f(x)$ as a polynomial in x for the following data by Newton's divided difference formula.

x	-4	-1	0	2	5
$f(x)$	1245	33	5	9	1335

Soln:-

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-4	1245				
		$\frac{33-1245}{(-1)-(-4)} = -404$			
-1	33		$\frac{-28-(-404)}{0-(-4)} = 94$		
		$\frac{5-33}{0-(-1)} = -28$		$\frac{10-94}{2-(-4)} = -14$	
0	5		$\frac{2-(-28)}{2-(-1)} = 10$		$\frac{13+14}{5-(-4)} = 3$
		$\frac{9-5}{2-0} = 2$		$\frac{88-10}{5-(-1)} = 13$	
2	9		$\frac{442-2}{5-0} = 88$		
		$\frac{1335-9}{5-2} = 442$			
5	1335				

By Newton's divided difference interpolation formula

$$f(x) = f(x_0) + (x-x_0)\, f(x_0, x_1) + (x-x_0)(x-x_1)\, f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2)\, f(x_0, x_1, x_2, x_3) + (x-x_0)(x-x_1)(x-x_2)(x-x_3)\, f(x_0, x_1, x_2, x_3, x_4)$$

Here $x_0 = -4 \quad x_1 = -1 \quad x_2 = 0 \quad x_3 = 2 \quad x_4 = 5$

$f(x_0) = 1245$

$f(x_0, x_1) = -404$

$f(x_0, x_1, x_2) = 94$

$f(x_0, x_1, x_2, x_3) = -14$

$f(x_0, x_1, x_2, x_3, x_4) = 3$

$$f(x) = 1245 + (x+4)(-404) + (x+4)(x+1)(94) + (x+4)(x+1)(x)(-14) + (x+4)(x+1)(x)(x-2)(3)$$

$$= 1245 - 404x - 1616 + 94[x^2+5x+4] - 14x[x^2+5x+4] + 3x[(x^2+5x+4)(x-2)]$$

$$= 1245 - 404x - 1616 + 94x^2 + 470x + 376 - 14x^3 - 70x^2 - 56x + 3x[x^3 - 2x + 5x^2 - 10x + 4x - 8]$$

$$= -14x^3 + 24x^2 + 10x + 5 + 3x[x^3 + 5x^2 - 8x - 8]$$

$$= -14x^3 + 24x^2 + 10x + 5 + 3x^4 + 15x^3 - 24x^2 - 24x = 3x^4 + x^3 - 14x + 5$$

(3) using Newton's divided difference formula find the missing value from the table.

x	1	2	4	5	6
y	14	15	5	-	9

Soln:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	14			
		$\frac{15-14}{2-1}=1$		
2	15		$\frac{-5-1}{4-1}=-2$	
		$\frac{5-15}{4-2}=-5$		$\frac{\frac{7}{4}+2}{6-1}=\frac{15/4}{5}=\frac{3}{4}$
4	5		$\frac{2+5}{6-2}=\frac{7}{4}$	
		$\frac{9-5}{6-4}=2$		
6	9			

$$f(x) = f(x_0) + (x-x_0)\,f(x_0,x_1) + (x-x_0)(x-x_1)\,f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)\,f(x_0,x_1,x_2,x_3) + \cdots$$

$$= 14 + (x-1)(1) + (x-1)(x-2)(-2) + (x-1)(x-2)(x-4)\left(\tfrac{3}{4}\right)$$

$$= 14 + x - 1 - 2(x-1)(x-2) + \tfrac{3}{4}(x-1)(x-2)(x-4)$$

$$f(5) = 13 + 5 - 2(4)(3) + \tfrac{3}{4}(4)(3)(1)$$

$$= 18 - 24 + 9 = 3$$

INTERPOLATION WITH A CUBIC SPLINE

Formula:

$$f_i(x) = \frac{f_i''(x_{i-1})}{6(x_i - x_{i-1})}(x_i - x)^3 + \frac{f_i''(x_i)}{6(x_i - x_{i-1})}(x - x_{i-1})^3$$
$$+ \left[\frac{f(x_{i-1})}{x_i - x_{i-1}} - \frac{f''(x_{i-1})(x_i - x_{i-1})}{6}\right](x_i - x)$$
$$+ \left[\frac{f(x_i)}{x_i - x_{i-1}} - \frac{f''(x_i)(x_i - x_{i-1})}{6}\right](x - x_{i-1}) \quad \text{--- (1)}$$

This equation contains only two unknowns - the second derivatives at the end of each interval.

These unknowns can be evaluated using the following equation.

$$(x_i - x_{i-1}) f''(x_{i-1}) + 2(x_{i+1} - x_{i-1}) f''(x_i) + (x_{i+1} - x_i) f''(x_{i+1})$$
$$= \frac{6}{x_{i+1} - x_i}\left[f(x_{i+1}) - f(x_i)\right] + \frac{6}{x_i - x_{i-1}}\left[f(x_{i-1}) - f(x_i)\right] \quad \text{--- (2)}$$

(1) From the following table.

x	x_{i-1} 1	x_i 2	x_{i+1} 3
y	-8	-1	18

Comput $y(1.5)$ and $y'(1)$ using cubic spline

Soln:

$$(x_i - x_{i-1}) f''(x_{i-1}) + 2(x_{i+1} - x_{i-1}) f''(x_i) + (x_{i+1} - x_i) f''(x_{i+1})$$
$$= \frac{6}{x_{i+1} - x_i}\left[f(x_{i+1}) - f(x_i)\right] + \frac{6}{x_i - x_{i-1}}\left[f(x_{i-1}) - f(x_i)\right] \quad \text{--- (2)}$$

$$(2-1) f''(1) + 2(3-1) f''(2) + (3-2) f''(3) = \frac{6}{(3-2)}(18+1) + \frac{6}{2-1}[-8+1]$$

$$f''(1) + 4f''(2) + f''(3) = 6(19) + 6(-7)$$

$f''(1) = 0 \quad f''(3) = 0$ at the end points, $f'(3) = 0$, $4f''(2) = 72$

$f''(2) = 18$

From (1) we get

$$f(x) = 0 + \frac{1}{6}\frac{f''(2)}{(2-1)}(x-1)^3 + \left[\frac{-8}{2-1} - 0\right](2-x) + \left[\frac{-1}{2-1} - \frac{18}{6}(2-1)\right](x-1)$$

$$= \frac{1}{6} 18(x-1)^3 + (-8)(2-x) + [-1-3][x-1]$$

$$= 3(x-1)^3 - 8(2-x) - 4(x-1)$$

$$= 3(x-1)^3 - 16 + 8x - 4x + 4$$

$$= 3[x^3 - 3x^2 - 3 + 1] - 16 + 4x + 4 = 3x^3 - 9x^2 + 13 - 15$$

$$= 3x^3 - 9x^2 - 8$$

$$y(1.5) = f(1.5) = 3(0.5)^3 + 4(1.5) - 12 = -\frac{45}{8}$$

$$y' = f'(x) = 9(x-1)^2 + 4$$

$$y'(1) = f'(1) = 4 \text{ Ans.}$$

Another method

$$S(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i - x)\left[y_{i-1} - \frac{h^2}{6} M_{i-1}\right]$$

$$+ \frac{1}{h}(x - x_{i-1})\left[y_i - \frac{h^2}{6} M_i\right] \cdots$$

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right] \text{ for } i = 1, 2, 3 \cdots (n-1)$$

① From the following table:

x	1	2	3
y	-8	-1	18

Compute $y(1.5)$ and $y'(1)$ using cubic spline

Soln:

Here $h = 1$, and $n = 2$. also assume $M_0 = 0$ and $M_2 = 0$

we have

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right] \text{ for } i = 1, 2 \cdots (n-1)$$

from this

$$M_0 + 4M_1 + M_2 = 6[y_0 - 2y_1 + y_2]$$

$$\therefore 4M_1 = 6[-8 - 2(-1) + 18] = 72$$

$$\therefore M_1 = 18$$

$$S(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i - x)\left[y_{i-1} - \frac{h^2}{6} M_{i-1}\right]$$

$$+ \frac{1}{h}(x - x_{i-1})\left[y_i - \frac{h^2}{6} M_i\right] \cdots ⑥$$

For $1 \le x \le 2$ putting $i = 1$ we get

$$S(x) = \frac{1}{6}\left[18(x-1)^3\right] + (2-x)(-8) - 4(x-1)$$

$$= 3(x-1)^3 + 4x - 12 = 3x^3 - 9x^2 + 13x - 15$$

13

$y(1.5) = S(1.5) = 3(0.5)^3 + 4(1.5) - 12 = -\frac{45}{8}$

$y' = S'(x) = 9(x-1)^2 + 4$

$y'(1) = 4.$

② Given the points $(0, 0)$ $(\pi/2, 1)$ and $(\pi, 0)$ satisfying the function $y = \sin x$ $(0 \le x \le \pi)$ determine the value of $y(\pi/6)$ using the cubic spline approximation.

x	0	$\pi/2$	π
y	0	1	0

Soln:

Here $h = \pi/2$, $n = 2$, also assume $M_0 = 0$, $M_2 = 0$

we have

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}] \text{ for } i = 1, 2, \ldots (n-1)$$

From this

$$M_0 + 4M_1 + M_2 = \frac{6}{(\pi/2)^2}[0 - 2 + 0] = \frac{-48}{\pi^2}$$

$$4M_1 = \frac{-48}{\pi^2}$$

$$M_1 = \frac{-48}{4\pi^2} = \frac{-12}{\pi^2}$$

In the interval $[0, \pi/2]$, the natural cubic spline is given by

$$S_1(x) = \frac{1}{6(\pi/2)}\left[(x-0)^3\left(\frac{-12}{\pi^2}\right) + \frac{1}{(\pi/2)}\left[\pi/2 - x\right]\left[0 - \frac{(\pi/2)^2}{6}0\right] + \frac{1}{(\pi/2)}[x-0]\left[1 - \frac{(\pi/2)^2}{6}\left(\frac{-12}{\pi^2}\right)\right]\right.$$

$$= \frac{1}{3\pi}\left[x^3\left(\frac{-12}{\pi^2}\right) + \frac{2}{\pi}(x)\left[1 + \frac{1}{2}\right]\right]$$

$$= \frac{1}{3\pi}\left[\frac{-12}{\pi^2}x^3\right] + \frac{3}{\pi}x$$

$$= \frac{1}{\pi}\left[\frac{-4}{\pi^2}x^3\right] + \frac{3}{\pi}x. = \frac{2}{\pi}\left[\frac{-2}{\pi^2}x^3 + \frac{3}{2}x\right]$$

$$y(\pi/6) = \frac{2}{\pi}\left[\frac{-\pi}{108} + \frac{\pi}{4}\right] = 0.4815$$

NEWTON FORWARD AND BACKWARD DIFFERENCE FORMULA

Forward interpolation formula

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

where $u = \frac{x - x_0}{h}$

(1) using Newton's forward interpolation formula, find the polynomial $f(x)$ satisfying the following data. Hence evaluate y at $x=5$.

x	4	6	8	10
y	1	3	8	10

Soln:
we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
(x_0) 4	(y_0) 1			
		$3-1=2(\Delta y_0)$		
			$5-2=3(\Delta^2 y_0)$	
(x_1) 6	(y_1) 3			$-3-3=-6(\Delta^3 y_0)$
		$8-3=5(\Delta y_1)$		
			$2-5=-3(\Delta^2 y_1)$	
(x_2) 8	(y_2) 8			
		$10-8=2(\Delta y_2)$		
(x_3) 10	(y_3) 10			

There are only 4 data given. Hence the polynomial will be degree 3.

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 .$$

where $u = \frac{x - x_0}{h}$ Here $x_0 = 4$, $h = 6-4 = 2$ [difference]

$$y(x) = 1 + \frac{\left(\frac{x-4}{2}\right)}{1!}(2) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)}{2!}(3) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)\left(\frac{x-8}{2}\right)}{3!}(-6)$$

$$= 1 + x - 4 + \frac{(x-4)(x-6)}{8}(3) + \frac{(x-4)(x-6)(x-8)}{(8)(6)}(-6).$$

$$= x - 3 + \frac{3}{8}(x-4)(x-6) - \frac{1}{8}(x-4)(x-6)(x-8)$$

$$= x - 3 + \frac{3}{8}[x^2 - 10x + 24] - \frac{1}{8}[(x^2 - 10x + 24)(x-8)]$$

$= x - 3 + \frac{3}{8}\left[x^2 - 10x + 24\right] - \frac{1}{8}\left[x^3 - 10x^2 + 24x - 8x^2 + 80x - 192\right]$

$= \frac{1}{8}\left[8x - 24 + x^2 - 10x + 24 - x^3 + 10x^2 - 24x + 8x^2 - 80x + 192\right]$

$= \frac{1}{8}\left[-x^3 + 19x^2 - 106x + 192\right]$

$y(5) = \frac{1}{8}\left(-(5)^3 + 19(5)^2 - 106(5) + 192\right]$

$= \frac{1}{8}\left[-125 + 475 - 530 + 192\right]$

$= \frac{1}{8}\left[12\right]$

$y(5) = 1.5.$

(2) A third degree polynomial passes through the points (0, -1) (1, 1) (2, 1) and (3, -2) using Newton's forward interpolation formula find the polynomial. Hence find the value at 1.5

Soln

we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
(x_0) 0	(y_0) -1	$1+1 = 2(\Delta y_0)$		
			$0-2 = -2(\Delta^2 y_0)$	
(x_1) 1	(y_1) 1			$-3+2 = -1(\Delta^3 y_0)$
		$1-1 = 0(\Delta y_1)$		
(x_2) 2	(y_2) 1		$-3-0 = -3(\Delta^2 y_1)$	
		$-2-1 = -3(\Delta y_2)$		
(x_3) 3	(y_3) -2			

There are only 4 data given. Hence the polynomial of degree 3.

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

where $u = \frac{x - x_0}{h}$

$x_0 = 0, \quad h = 1 - 0 = 1$ (difference)

$\therefore u = x$

$$y(x) = -1 + \frac{x}{1!}(2) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(-1)$$

$= -1 + 2x - x(x-1) - \frac{1}{6} x(x-1)(x-2)$

$= -1 + 2x - x^2 + x - \frac{1}{6} x [x^2 - 3x + 2]$

$= -x^2 + 3x - 1 - \frac{1}{6} [x^3 - 3x^2 + 2x]$

$= \frac{1}{6} [-6x^2 + 18x - 6 - x^3 + 3x^2 - 2x]$

$= \frac{1}{6} [-x^3 - 3x^2 + 16x - 6]$

$= -\frac{1}{6} [x^3 + 3x^2 - 16x + 6]$

$y(1.5) = -\frac{1}{6} [(1.5)^3 + 3(1.5)^2 - 16(1.5) + 6]$

$= -\frac{1}{6} [3.375 + 6.75 - 24 + 6]$

$= -\frac{1}{6} [-7.875]$

$y(1.5) = 1.3125$

(3) From the data given below, find the number of students whose weight is between 60 to 70.

Weight in lbs:	0-40	40-60	60-80	80-100	100-120
No. of students:	250	120	100	70	50

Soln:

Difference table

x weight	y (No. of Students)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
Below 60	370		-20		
		100		-10	
Below 80	470				20
			-30		
		70		10	
Below 100	540				
			-20		
		50			
Below 120	590				

Let us calculate the number of students whose weight is less than 70.

We will use forward difference formula.

$$u = \frac{x - x_0}{h} = \frac{70 - 40}{20} = 1.5$$

$$y(70) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \cdots$$

$$= 250 + (1.5)(120) + \frac{(1.5)(0.5)}{2}(-20) + \frac{(1.5)(0.5)(-0.5)}{6}(-10)$$

$$+ \frac{(1.5)(0.5)(-0.5)(-1.5)}{24}(20)$$

$$= 250 + 180 - 7.5 + 0.625 + 0.46875$$

$$= 423.59$$

$$\simeq 424.$$

Number of students whose weight is between 60 and 70.

$$= y(70) - y(60) = 424 - 370 = 54.$$

Newton's backward interpolation formula

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \cdots$$

where $v = \frac{x - x_n}{h}$

① Use Newton's backward difference formula to construct an interpolating polynomial of degree 3 for the data.

$f(-0.75) = -0.07181250$, $f(-0.5) = -0.024750$

$f(-0.25) = 0.33493750$, $f(0) = 1.10100$. Hence find $f(-\frac{1}{3})$

Soln:-

Newton's backward difference formula is

$$y(x) = y_3 + \frac{v}{1!}\nabla y_3 + \frac{v(v+1)}{2!}\nabla^2 y_3 + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_3$$

where $v = \frac{x - x_3}{h}$

Here we for the difference table.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
(x_0) (-0.75)	y_0 -0.07181250			
		0.0470625		
(x_1) -0.5	y_1 -0.024750		0.312625	
		0.3596875		$(\nabla^3 y_3)$ 0.09375
(x_2) -0.25	y_2 0.33493750		$(\nabla^2 y_3)$ 0.400375	
		(∇y_3) 0.7660625		
(x_3) 0	y_3 1.10100			

Here $x_3 = 0$ $\quad h = 0.25 \quad v = \frac{x}{0.25} = \frac{x}{(1/4)} = 4x$

$$y(x) = 1.10100 + 4x(0.7660625) + \frac{4x(4x+1)}{2}(0.406375) + \frac{4x(4x+1)(4x+2)}{6}(0.09375)$$

$$= 1.10100 + 3.06425 + 0.81275x(4x+1) + 0.0625x(4x+1)(4x+2)$$

$$= 1.101 + 3.06425x^2 + 0.81275x + 0.0625x[16x^2 + 12x + 2]$$

$$= 1.101 + 3.06425x + 3.251x^2 + 0.81275x + x^3 + 0.75x^2 + 0.125x.$$

$$= x^3 + 4.001x^2 + 4.002x + 1.101$$

To find $f(-1/3)$.

$$y(-1/3) = (-1/3)^3 + (4.001)(-1/3)^2 + 4.002(-1/3) + 1.101$$

$$= -\frac{1}{27} + 4.001(1/9) - 4.002(1/3) + 1.101$$

$$= 0.174518518 \text{ Ans}$$

② From the following table find the value of $\tan(0.28)$

x	0.10	0.15	0.20	0.25	0.30
$y = \tan x$	0.1003	0.1511	0.2027	0.2533	0.3093

Soln: Let us form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

Since 0.28 lies in the end of the table, let us use Newton's backward interpolation formula.

$$f(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \dots$$

where $v = \dfrac{x - x_n}{h} = \dfrac{0.28 - 0.30}{0.05} = -0.04 \quad [\because x_n = 0.30]$

$$y = 0.3093 + \frac{(-0.4)}{1!}(0.0540) + \frac{(-0.4)(-0.4+1)}{2!}(0.0014)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)}{3!}(0.0004)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)}{4!}(0.0002)$$

$$y = 0.309 - 0.0216 - 0.000168 - 0.0000256 - 0.00000832$$

$$y = 0.28720.$$

Numerical Differentiation and Integration

Derivatives from Difference tables - divided differences and Finite Differences

Newton's forward difference formula

Newton's forward difference interpolation formula is

$$y(x_0+uh) = y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

where $y(x)$ is a polynomial of degree n in x and $u = \frac{x-x_0}{h}$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

Newton's backward difference formula

Newton's backward difference formula is

$$y(x) = y(x_n+vh) = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

where $v = \frac{x-x_0}{h}$

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right]$$

Derivative using Stirling formula

The Stirling's formula is

$$y(x) = y_0 + \frac{u}{2}\left[\Delta y_0 + \Delta y_{n-1}\right] + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u^3-u}{12}\left[\Delta^3 y_{-1} + \Delta^3 y_{-2}\right] + \frac{u^4-u^2}{24}\Delta^4 y_{-2} + \cdots$$