MA6459 NUMERICAL METHODS L T P C 3 1 0 4

OBJECTIVES:

This course aims at providing the necessary basic concepts of a few numerical methods and give procedures for solving numerically different kinds of problems occurring in engineering and technology

UNIT I SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS 10+3

Solution of algebraic and transcendental equations - Fixed point iteration method – Newton Raphson method- Solution of linear system of equations - Gauss elimination method – Pivoting - Gauss Jordan method – Iterative methods of Gauss Jacobi and Gauss Seidel - Matrix Inversion by Gauss Jordan method - Eigen values of a matrix by Power method.

UNIT II INTERPOLATION AND APPROXIMATION 8+3

Interpolation with unequal intervals - Lagrange's interpolation – Newton’s divided difference interpolation – Cubic Splines - Interpolation with equal intervals - Newton’s forward and backward difference formulae.

UNIT III NUMERICAL DIFFERENTIATION AND INTEGRATION 9+3

Approximation of derivatives using interpolation polynomials - Numerical integration using Trapezoidal, Simpson’s 1/3 rule – Romberg’s method - Two point and three point Gaussian quadrature formulae – Evaluation of double integrals by Trapezoidal and Simpson’s 1/3 rules.

UNIT IV INITIAL VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS 9+3

Single Step methods - Taylor’s series method - Euler’s method - Modified Euler’s method – Fourth order Runge-Kutta method for solving first order equations - Multi step methods - Milne’s and Adams-Bash forth predictor corrector methods for solving first order equations.

UNIT V BOUNDARY VALUE PROBLEMS IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS 9+3

Finite difference methods for solving two-point linear boundary value problems - Finite difference techniques for the solution of two dimensional Laplace’s and Poisson’s equations on rectangular domain – One dimensional heat flow equation by explicit and implicit (Crank Nicholson) methods –One dimensional wave equation by explicit method.

TOTAL (L:45+T:15): 60 PERIODS

OUTCOMES:

The students will have a clear perception of the power of numerical techniques, ideas and would be able to demonstrate the applications of these techniques to problems drawn from industry, management and other engineering fields.

TEXT BOOKS:

1. Grewal. B.S., and Grewal. J.S.,"Numerical methods in Engineering and Science", Khanna Publishers, 9th Edition, New Delhi, 2007.
2. Gerald. C. F., and Wheatley. P. O., "Applied Numerical Analysis", Pearson Education, Asia,
6th Edition, New Delhi, 2006.

REFERENCES:

1. Chapra. S.C., and Canale.R.P., "Numerical Methods for Engineers, Tata McGraw Hill, 5th Edition, New Delhi, 2007.
2. Brian Bradie. "A friendly introduction to Numerical analysis", Pearson Education, Asia, New Delhi, 2007.
3. Sankara Rao. K., "Numerical methods for Scientists and Engineers", Prentice Hall of India Private, 3rd Edition, New Delhi, 2007.

SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS

Method of False position (or) Regula Falsi method (or) linear interpolation method.

If $f(x_1)f(a) < 0$, then x_2 lies between x_1 and a

$$x_2 = \frac{a f(x_1) - x_1 f(a)}{f(x_1) - f(a)}$$

① Find the positive root of $x^3 - 2x - 5 = 0$ by the Regula Falsi method.

Soln:

Let $f(x) = x^3 - 2x - 5 = 0$

There is only one poistive root by Descarte's rule of signs

$f(0) = -5 = -ve$

$f(1) = 1-2-5 = -6 = -ve$

$f(2) = 8-4-5 = -1 = -ve$

$f(3) = 27-6-5 = 16 = +ve$

Therefore the positive root lies between 2 and 3. It is closer to 2 also.

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$= \frac{2 f(3) - 3 f(2)}{f(3) - f(2)}$$

$$= \frac{2(16) - 3(-1)}{16 - (-1)}$$

$$= \frac{32 + 3}{17} = \frac{35}{17}$$

$= 2.0588$ [correct to 4 decimal places]

$f(x_1) = f(2.0588) = (2.0588)^3 - 2(2.0588) - 5$

$= 8.7265 - 4.1176 - 5$

$= -0.3911$

$\therefore$ The root lies between 2.0588 and 3

$$x_2 = \frac{2.0588\, f(3) - 3\, f(2.0588)}{f(3) - f(2.0588)}$$

$$= \frac{2.0588\,(16) - 3(-0.3911)}{16 - (-0.3911)}$$

$$= \frac{32.9408 + 1.1733}{16.3911} = \frac{34.1141}{16.3911}$$

$$= 2.0813$$

$$f(x_2) = f(2.0813) = (2.0813)^3 - 2(2.0813) - 5$$

$$= 9.0158 - 4.1626 - 5$$

$$= -0.1468$$

$\therefore$ The root lies between 2.0813 and 3

$$x_3 = \frac{2.0813\, f(3) - 3\, f(2.0813)}{f(3) - f(2.0813)}$$

$$= \frac{2.0813(16) - 3(-0.1468)}{16 - (-0.1468)}$$

$$= \frac{33.3008 + 0.4404}{16.1468} = \frac{33.7412}{16.1468}$$

$$= 2.08965$$

$$= 2.0897 \text{ (four decimal places)}$$

$$f(x_3) = f(2.0897) = (2.0897)^3 - 2(2.0897) - 5$$

$$= -0.054 = -ve$$

$\therefore$ The root lies between 2.0897 and 3

$$x_4 = \frac{(2.0897)\, f(3) - 3\, f(2.0897)}{f(3) - f(2.0897)}$$

$$= \frac{(2.0897)(16) - 3(-0.054)}{16 - (-0.054)}$$

$$= \frac{33.4352 + 0.162}{16.054}$$

$$= \frac{33.5972}{16.054} = 2.0928$$

$$f(x_4) = f(2.0928) = (2.0928)^3 - 2(2.0928) - 5$$

$$= 9.1661 - 4.1856 - 5$$

$$= -0.0195 = -ve$$

$\therefore$ The root lies between 2.0928 and 3

$$x_5 = \frac{2.0928\, f(3) - 3\, f(2.0928)}{f(3) - f(2.0928)}$$

$$= \frac{(2.0928)(16) - 3(-0.0195)}{16 - (-0.0195)}$$

$$= \frac{33.4848 + 0.0585}{16.0195}$$

$$= \frac{33.5433}{16.0195}$$

$$= 2.0939$$

$$f(x_5) = f(2.0939) = (2.0939)^3 - 2(2.0939) - 5$$

$$= 9.1805 - 4.1878 - 5$$

$$= -0.0073$$

$$= -ve$$

$\therefore$ The root lies between 2.0939 and 3

$$x_6 = \frac{(2.0939)\, f(3) - 3\, f(2.0939)}{f(3) - f(2.0939)}$$

$$= \frac{(2.0939)(16) - 3(-0.0073)}{16 - (-0.0073)}$$

$$= \frac{33.5024 + 0.0219}{16.0073} = \frac{33.5243}{16.0073} = 2.0943$$

$f(x_6) = f(2.0943) = (2.0943)^3 - 2(2.0943) - 5$

$= 9.1868 - 4.1886 - 5$

$= -0.0028$

$= -ve$

$\therefore$ The root lies between 2.0943 and 3

$$x_7 = \frac{(2.0943)f(3) - 3f(2.0943)}{f(3) - f(2.0943)}$$

$$= \frac{(2.0943)(16) - 3(-0.0028)}{16 - (-0.0028)}$$

$$= \frac{33.5088 + 0.0084}{16.0028}$$

$$= \frac{33.5172}{16.0028} = 2.0945$$

$f(x_7) = f(2.0945) = (2.0945)^3 - 2(2.0945) - 5$

$= 9.1884 - 4.189 - 5$

$= -0.0006 = -ve$

$\therefore$ The root lies between 2.0945 and 3

$$x_8 = \frac{2.0945\, f(3) - 3f(2.0945)}{f(3) - f(2.0945)}$$

$$= \frac{(2.0945)(16) - 3(-0.0006)}{16 - (-0.0006)}$$

$$= \frac{33.512 + 0.0018}{16.0006}$$

$$= \frac{33.5138}{16.0006}$$

$= 2.0945$

We observe that $x_7 = x_8 = 2.0945$ correct to 4 places of decimals.

Hence the required root correct to four places of decimals is 2.0945

The results of the complete working are tabulated below.

Iteration(r)	a	b	x_r	sign of $f(x_r)$
1	2	3	2.0588	-0.3911
2	2.0588	3	2.0813	-0.1468
3	2.0813	3	2.0897	-0.054
4	2.0897	3	2.0928	-0.0195
5	2.0928	3	2.0939	-0.0073
6	2.0939	3	2.0943	-0.0028
7	2.0943	3	2.0945	-0.0006
8	2.0945	3	2.0945	

Formula $x_r = \dfrac{a f(b) - b f(a)}{f(b) - f(a)}$

We observe that $x_7 = x_8 = 2.0945$

Hence the required root is 2.0945

② Using method of false position find a root of the equation $x^3 - 3x - 5 = 0$

Soln:-

Given $f(x) = x^3 - 3x - 5$

$f(0) = 0 - 0 - 5 = -5 = -ve$

$f(1) = 1 - 3 - 5 = 1 - 8 = -7 = -ve$

$f(2) = 8 - 6 - 5 = 8 - 11 = -3 = -ve$

$f(3) = 27 - 9 - 5 = 27 - 14 = 13 = +ve$

$\therefore$ one root lies between 2 and 3

let $a = 2, \quad b = 3$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$= \frac{2(13) - 3(-3)}{13 - (-3)}$$

$$= \frac{26+9}{16} = \frac{35}{16} = 2.1875$$

$$f(x_1) = f(2.1875) = (2.1875)^3 - 3(2.1875) - 5$$

$$= 10.4675 - 6.5625 - 5$$

$$= -1.095$$

$\therefore$ one root lies between 2.1875 and 3

$$x_2 = \frac{(2.1875) f(3) - 3 f(2.1875)}{f(3) - f(2.1875)}$$

$$= \frac{(2.1875)(13) - 3(-1.095)}{13 - (-1.095)}$$

$$= \frac{28.4375 + 3.285}{14.095}$$

$$= \frac{31.7225}{14.095} = 2.2506$$

$$f(x_2) = f(2.2506) = (2.2506)^3 - 3(2.2506) - 5$$

$$= 11.3997 - 6.7518 - 5$$

$$= -0.3521 = -ve$$

$\therefore$ The root lies between 2.2506 and 3

$$x_3 = \frac{2.2506 f(3) - 3 f(2.2506)}{f(3) - f(2.2506)}$$

$$= \frac{(2.2506)(13) - 3(-0.3521)}{13 - (-0.3521)}$$

$$= \frac{29.2578 + 1.0563}{13.3521} = \frac{30.3141}{13.3521} = 2.2704$$

$$f(x_3) = f(2.2704) = (2.2704)^3 - 3(2.2704) - 5$$

$$= 11.7033 - 6.8112 - 5$$

$$= -0.1079 = -ve$$

$\therefore$ The root lies between 2.2704 and 3

$$x_4 = \frac{2.2704\, f(3) - 3 f(2.2704)}{f(3) - f(2.2704)}$$

$$= \frac{(2.2704)(13) - 3(-0.1079)}{13 - (-0.1079)}$$

$$= \frac{29.5152 + 0.3237}{13.1079} = \frac{29.8389}{13.1079} = 2.2764$$

$$f(x_4) = f(2.2764) = (2.2764)^3 - 3(2.2764) - 5$$

$$= 11.7963 - 6.8292 - 5$$

$$= -0.0329 = -ve$$

$\therefore$ The root lies between 2.2764 and 3

$$x_5 = \frac{(2.2764)\, f(3) - 3 f(2.2764)}{f(3) - f(2.2764)}$$

$$= \frac{(2.2764)(13) - 3(-0.03929)}{13 - (-0.0329)}$$

$$= \frac{25.5932 + 0.0987}{13.0329} = \frac{29.6919}{13.0329} = 2.2782$$

$$f(x_5) = f(2.2782) = (2.2782)^3 - 3(2.2782) - 5$$

$$= 11.8243 - 6.8346 - 5$$

$$= -0.0103 = -ve$$

$\therefore$ The root lies between 2.2782 and 3

$$x_6 = \frac{(2.2782)\, f(3) - 3 f(2.2782)}{f(3) - f(2.2782)}$$

$$= \frac{(2.2782)(13) - 3(-0.0103)}{13 - (-0.0103)}$$

$$= \frac{29.6166 + 0.0309}{13.0103} = \frac{29.6475}{13.0103} = 2.2788$$

$$f(x_6) = f(2.2788) = (2.2788)^3 - 3(2.2788) - 5$$

$$= 11.8336 - 6.8304 - 5$$

$$= -0.0028 = -ve$$

$\therefore$ The root lies between 2.2788 and 3

$$x_7 = \frac{2.2788\, f(3) - 3 f(2.2788)}{f(3) - f(2.2788)}$$

$$= \frac{(2.2788)(13) - 3(-0.0028)}{13 - (-0.0028)}$$

$$= \frac{29.6244 + 0.0084}{13.0028}$$

$$= \frac{29.6328}{13.0028}$$

$$= 2.2790$$

$$f(x_7) = f(2.2790) = (2.2790)^3 - 3(2.2790) - 5$$

$$= 11.8367 - 6.837 - 5$$

$$= -0.0003 = -ve$$

$\therefore$ The root lies between 2.279 and 3

$$x_8 = \frac{2.279\, f(3) - 3 f(2.279)}{f(3) - f(2.279)}$$

$$= \frac{(2.279)(13) - 3(-0.0003)}{13 - (-0.0003)}$$

$$= \frac{29.627 + 0.0009}{13.0003}$$

$$= \frac{29.6279}{13.0003}$$

$$= 2.2790$$

we observe that $x_7 = x_8 = 2.2790$ correct to four places of decimals.

Hence the required root is 2.2790.

TABLE

$f(x) = x^3 - 3x - 5$	Formula $x_r = \frac{af(b) - bf(a)}{f(b) - f(a)}$			
Iteration(r)	a	b	x_r	$f(x_r)$
1	2	3	2.1875	
2	2.1875	3	2.2506	-1.095
3	2.2506	3	2.2704	-0.3521
4	2.2704	3	2.2764	-0.1079
5	2.2764	3	2.2782	-0.0329
6	2.2782	3	2.2788	-0.0103
7	2.2788	3	2.2790	-0.0028
8	2.2790	3	2.2790	-0.0003

Hence the required root is 2.2790

(3) Find an approximate root of $x \log_{10} x - 1.2 = 0$ by Regula Falsi method.

Soln:- Let $f(x) = x \log_{10}^{x} - 1.2$

$f(1) = 0 - 1.2 = -1.2 = -ve$

$f(2) = 2(0.30103) - 1.2 = -0.5979 = -ve$

$f(3) = 3 \cdot (0.47712) = 0.2314 = +ve$

Hence a root lies between 2 and 3

$$x_1 = \frac{2f(3) - 3f(2)}{f(3) - f(2)}$$

$$= \frac{2(0.2314) - 3(-0.5979)}{0.2314 - (-0.5979)} = \frac{0.4628 + 1.7937}{0.8293}$$

$$= \frac{2.2565}{0.8293} = 2.7210$$

$$f(x_1) = f(2.7210) = (2.7210)\log_{10} 2.7210 - 1.2$$

$$= -0.0171 = -ve$$

Therefore the root lies between 2.7210 and 3

$$x_2 = \frac{(2.7210)\,f(3) - 3\,f(2.7210)}{f(3) - f(2.7210)}$$

$$= \frac{0.68094}{0.2485} = 2.7402$$

$$f(x_2) = f(2.7402) = (2.7402)\log_{10} 2.7402 - 1.2$$

$$= -0.0004 = -ve$$

$\therefore$ The root lies between 2.7402 and 3

$$x_3 = \frac{2.7402\,f(3) - 3\,f(2.7402)}{f(3) - f(2.7402)}$$

$$= \frac{(2.7402)(0.2314) - 3(-0.0004)}{(0.2314) - (-0.0004)}$$

$$= \frac{(2.7407)(0.2314) - 3(-0.0004)}{(0.2314) - (-0.0004)}$$

$$= 2.7407$$

$$f(x_3) = f(2.7407) = 2.7407\log_{10} 2.7407 - 1.2$$

$$= 0.0001 = +ve$$

$\therefore$ The root lies between 2.7402 and 2.7407

$$x_4 = \frac{2.7402\,f(2.7407) - (2.7407)\,f(2.7402)}{f(2.7407) - f(2.7402)}$$

$$= \frac{(2.7402)(0.0001) - (2.7407)(-0.0004)}{(0.0001) - (-0.0004)}$$

$$= \frac{(2.7402)(0.0001) + (2.7407)(0.0004)}{0.0001 + 0.0004}$$

$$= 2.7406$$

$$f(x_4) = f(2.7406) = 2.7406 \log_{10} 2.7406 - 1.2$$

$$= -0.0004 = -ve$$

$$x_5 = \frac{2.7406\, f(2.7407) - 2.7407\, f(2.7406)}{f(2.7407) - f(2.7406)}$$

$$= \frac{(2.7406)(0.0001) - (2.7407)(-0.00004)}{0.0001 - (-0.00004)}$$

$$= \frac{(2.7406)(0.0001) + (2.7407)(0.00004)}{0.0001 + 0.00004} = 2.7406$$

we observe that $x_4 = x_5 = 2.7406$ correct to four places of decimals

Hence the required root is 2.7406

$f(x) = x \log_{10} x - 1.2$		Formula $x_r = \frac{a f(b) - b f(a)}{f(b) - f(a)}$		
Iteration(r)	a	b	x_r	$f(x_r)$
1	2	3	2.7210	-0.0171
2	2.7210	3	2.7402	-0.0004
3	2.7402	3	2.7407	0.0001
4	2.7402	2.7407	2.7406	-0.00004
5	2.7406	2.7407	2.7406	-

we find that $f(2.7406)$ is approaching zero

Hence the required root is 2.7406.

NEWTON'S METHOD [Newton - Raphson method]

Formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2 \ldots$$

① Find the positive root of $x^4 - x = 10$ correct to three decimal places using Newton - Raphson method.

Soln:

let $f(x) = x^4 - x - 10$

$f'(x) = 4x^3 - 1$

$f(1) = 1 - 1 - 10 = -10 = -ve$

$f(2) = 2^4 - 2 - 10 = 16 - 2 - 10 = 4$ +ve

∴ a root lies between 1 and 2.

Take $x_0 = 2$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \left[\frac{2^4 - 2 - 10}{4(2)^3 - 1}\right]$$

$$= 2 - \left[\frac{4}{31}\right]$$

$$= 1.8709$$

$= 1.871$ [correct to three decimal places]

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.871 - \frac{f(1.871)}{f'(1.871)}$$

$$= 1.871 - \frac{(1.871)^4 - 1.871 - 10}{4(1.871)^3 - 1}$$

$$= 1.871 - \frac{0.3835}{25.199} = 1.856$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.856 - \frac{f(1.856)}{f'(1.856)}$$

$$= 1.856 - \left[\frac{(1.856)^4 - 1.856 - 10}{4(1.856)^3 - 1}\right]$$

$$= 1.856 - \frac{0.010}{24.574}$$

$$= 1.856$$

The better approximate root is 1.856

(2) using Newton's iterative method find the root between 0 and 1 of $x^3 = 6x - 4$ correct to two decimal places.

Soln Let $f(x) = x^3 - 6x + 4$

$f'(x) = 3x^2 - 6$

$f(0) = 4 = +ve$

$f(1) = 1 - 6 + 4 = -1 = -ve$

$\therefore$ a root lies between 0 and 1

$|f(0)| > |f(1)|$

$\therefore$ This root is nearer to 1

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let $x_0 = 1$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{f(1)}{f'(1)} = 1 - \left[\frac{(1)^3 - 6(1) + 4}{3(1)^2 - 6}\right]$$

$$= 1 \frac{-(-1)}{-3} = 1 - \frac{1}{3} = 0.666$$

$= 0.67$ [Correct to two decimal places]

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.67 - \frac{f(0.67)}{f'(0.67)}$$

$$= 0.67 - \left[\frac{(0.67)^3 - 6(0.67) + 4}{3(0.67)^2 - 6}\right]$$

$$= 0.67 - \frac{0.28}{-4.65}$$

$$= 0.67 + \frac{0.28}{4.6\cancel{5}}$$

$$= 0.73$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.73 - \frac{f(0.73)}{f'(0.73)}$$

$$= 0.73 - \left[\frac{(0.73)^3 - 6(0.73) + 4}{3(0.73)^2 - 6}\right]$$

$$= 0.73 - \left[\frac{0.009}{-4.4013}\right]$$

$$= 0.73 + \frac{0.009}{4.4013}$$

$$= 0.7320$$

$= 0.73$ [correct to two decimal places]

Here $x_2 = x_3 = 0.73$

$\therefore$ The root is 0.73 correct to two decimal places

(3) Find the real positive root of $3x - \cos x - 1 = 0$ by Newton's method correct to 6 decimal places.

Soln

let $f(x) = 3x - \cos x - 1$

$f(0) = 0 - 1 - 1 = -2 = -ve$

$f(1) = 3 - \cos 1 - 1 = 2 - \cos 1 = 1.459698 = +ve$

$\therefore$ a root lies between 0 and 1

$|f(0)| > |f(1)|$

Hence the root is nearer to 1.

Let $x_0 = 0.6$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.6 - \frac{f(0.6)}{f'(0.6)}$$

$$= 0.6 - \left[\frac{3(0.6) - \cos(0.6) - 1}{3 + \sin(0.6)}\right]$$

$$= 0.6 - (-0.007101)$$

$$= 0.607108$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.607108 - \left[\frac{3(0.607108) - \cos(0.607108) - 1}{3 + \sin(0.607108)}\right]$$

$$= 0.607108 - (0.000006)$$

$$= 0.607102$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.607102 - \frac{f(0.607102)}{f'(0.607102)}$$

$$= 0.607102 - \left[\frac{3(0.607102) - \cos(0.607102) - 1}{3 + \sin(0.607102)}\right]$$

$$= 0.607102 - 0.0000004$$

$$= 0.607102$$

Here $x_2 = x_3 = 0.607102$

$\therefore$ The root is 0.607102 correct to six decimals

(4) Find by NR method, the root of $x \log_{10} x = 12.34$ Start with $x_0 = 10$.

Soln:

let $f(x) = x \log_{10} x - 12.34$

$$f'(x) = x \, \frac{1}{x} \log_{10} e + \log_{10} x$$

$$= \log_{10} e + \log_{10} x$$

Given $x_0 = 10$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 10 - \frac{f(10)}{f'(10)}$$

$$= 10 - \left[\frac{10 \log_{10} 10 - 12.34}{\log_{10} e + \log_{10} 10}\right]$$

$$= 10 - \left[\frac{-2.34}{1.4343}\right]$$

$$= 10 + \frac{2.34}{1.4343}$$

$$= 11.6315$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 11.6315 - \frac{f(11.6315)}{f'(11.6315)}$$

$$= 11.6315 - \left[\frac{11.6315 \log_{10} 11.6315 - 12.34}{\log_{10} e + \log_{10} 11.6315}\right]$$

$$= 11.6315 - \frac{0.0549}{1.5}$$

$$= 11.5949$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 11.5949 - \frac{f(11.5949)}{f'(11.5949)}$$

$$= 11.5949 - \left[\frac{11.5949 \log_{10} 11.5949 - 12.34}{\log_{10} e + \log_{10} 11.5949}\right]$$

$$= 11.5949 - \frac{0.00006}{1.4986}$$

$$= 11.5949$$

From x_2 and x_3 we find out the root is 11.5949.

FIXED POINT ITERATION $x = g(x)$ method.

(1) Solve the equation $x^2 - 2x - 3 = 0$ for the positive root by iteration method

Soln:

let $f(x) = x^2 - 2x - 3 = 0$

$f(x)$ is easy to factor to show roots at $x = -1$ and $x = 3$

Rearrange equation (1)

$$x = g(x) = \sqrt{2x+3}$$

let $x_0 = 4$

$$x_1 = g(x_0) = \sqrt{2x_0+3} = \sqrt{8+3} = \sqrt{11} = 3.31662$$

$$x_2 = g(x_1) = \sqrt{2x_1+3} = \sqrt{9.63325} = 3.10375$$

$$x_3 = g(x_2) = \sqrt{2x_2+3} = \sqrt{9.20750} = \sqrt{9.20750} = 3.03439$$

$$x_4 = g(x_3) = \sqrt{2x_3+3} = \sqrt{9.06877} = 3.01144$$

$$x_5 = g(x_4) = \sqrt{2x_4+3} = \sqrt{9.02288} = 3.00381$$

$$x_6 = g(x_6) = \sqrt{2x_6+3} = 3.00127$$

$$x_7 = g(x_7) = \sqrt{2x_7+3} = 3.00042$$

$x_8 = g(x_7) = \sqrt{2x_7+3} = 3.00014$

$x_9 = g(x_8) = \sqrt{2x_8+3} = 3.00005$

$x_{10} = g(x_9) = \sqrt{2x_9+3} = 3.00002$

$x_{11} = g(x_{10}) = \sqrt{2x_{10}+3} = 3.00001$

$x_{12} = g(x_{11}) = \sqrt{2x_{11}+3} = 3.00000$

$x_{13} = g(x_{12}) = \sqrt{2x_{12}+3} = 3.00000$

Here $x_{12} = x_{13} = 3$ [correct 5 decimal places]

Hence the root is 3.

② Find a real root of the equation $x^3+x^2-100=0$

Soln:-

Let $f(x) = x^3+x^2-100=0$

$f(0) = -100 = -ve$

$f(1) = 1+1-100 = -98 = -ve$

$f(2) = 8+4-100 = -88 = -ve$

$f(3) = 27+9-100 = -64 = -ve$

$f(4) = 64+16-100 = -20 = -ve$

$f(5) = 125+25-100 = 50 = +ve$

So there is a real root between 4 and 5

The given equation can be written as

$$x^2(x+1) = 100$$

$$x = \frac{10}{\sqrt{x+1}} = g(x)$$

$$g'(x) = \frac{10\left[-\frac{1}{2}\right]}{(x+1)^{3/2}} = \frac{-5}{(x+1)^{3/2}}$$

$$|g'(x)| = \frac{5}{(x+1)^{3/2}}$$

$$|g'(4)| = \frac{5}{5^{3/2}} < 1$$

$$|g'(5)| = \frac{5}{6^{3/2}} < 1$$

$\therefore$ $|g'(x)|$ is less than 1 in the interval $(4, 5)$

So the method can be applied

Let $x_0 = 4.2$

$$x_1 = g(x_0) = \frac{10}{\sqrt{x_0+1}} = \frac{10}{\sqrt{4.2+1}} = 4.38529$$

$$x_2 = g(x_1) = \frac{10}{\sqrt{x_1+1}} = \frac{10}{\sqrt{4.38529+1}} = 4.30919$$

$$x_3 = g(x_2) = \frac{10}{\sqrt{x_2+1}} = \frac{10}{\sqrt{4.30919+1}} = 4.33996$$

$$x_4 = g(x_3) = \frac{10}{\sqrt{x_3+1}} = \frac{10}{\sqrt{4.33996+1}} = 4.32744$$

$$x_5 = g(x_4) = \frac{10}{\sqrt{x_4+1}} = 4.33252$$

$$x_6 = g(x_5) = \frac{10}{\sqrt{x_5+1}} = 4.33046$$

$$x_7 = g(x_6) = \frac{10}{\sqrt{x_6+1}} = 4.33129$$

$$x_8 = g(x_7) = \frac{10}{\sqrt{x_7+1}} = 4.33096$$

$$x_9 = g(x_8) = \frac{10}{\sqrt{x_8+1}} = 4.33109$$

$$x_{10} = g(x_9) = \frac{10}{\sqrt{x_9+1}} = 4.33104$$

$$x_{11} = g(x_{10}) = \frac{10}{\sqrt{x_{10}+1}} = 4.33106$$

$$x_{12} = g(x_{11}) = \frac{10}{\sqrt{x_{11}+1}} = 4.33105$$

$$x_{13} = g(x_{12}) = \frac{10}{\sqrt{x_{12}+1}} = 4.33105$$

Here $x_{12} = x_{13} = 4.33105$ correct to 5 decimal places.

Gaussian Elimination method. & Gauss-Jordan method

① Solve the system of equations by (i) Gauss elimination method (ii) Gauss-Jordan method.

$$10x - 2y + 3z = 23$$
$$2x + 10y - 5z = -33$$
$$3x - 4y + 10z = 41$$

Soln:

(i) Gauss elimination method.

The given system is equivalent to

$$\begin{bmatrix} 10 & -2 & 3 \\ 2 & 10 & -5 \\ 3 & -4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 23 \\ -33 \\ 41 \end{bmatrix}$$

Here $[A, B] = \begin{bmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{bmatrix}$

Now, we will make the matrix A as a upper triangular

Fix the first row, change 2 and 3 row with row 1

$$[A, B] \sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & -32 & 91 & 341 \end{bmatrix} \quad \begin{matrix} R_2 \leftrightarrow 5R_2 - R_1 \\ R_3 \leftrightarrow 10R_3 - 3R_1 \end{matrix}$$

Fix 1 and 2 row, change 3 row with 2nd row

$$\sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{bmatrix} \quad R_3 \leftrightarrow 52R_3 + 34R_2$$

This is an upper triangular matrix

From (1) we get [by back substitution]

$$3780\,z = 11340$$

$$\boxed{z = 3}$$

$$52y - 28z = -188$$

$$52y - 28(3) = -188$$

$$y = -2$$

$$10x - 2y + 3z = 23$$

$$10x - 2(-2) + 3(3) = 23$$

$$10x + 4 + 9 = 23$$

$$10x + 13 = 23$$

$$10x = 23 - 13$$

$$10x = 10$$

$$x = 1$$

Hence the solution is $x = 1,\ y = -2,\ z = 3$

(ii) <u>Gauss - Jordan method.</u>

Take the equation (1)

$$(A, B) \sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{bmatrix} \quad \begin{matrix} R_1 \leftrightarrow 1260R_1 - R_3 \\ R_2 \leftrightarrow 135R_2 + R_3 \end{matrix}$$

Now we will make the matrix A a diagonal matrix

$\boxed{\text{Fix the third row and change 2nd row and first row}}$

$$\sim \begin{bmatrix} 12600 & -2520 & 0 & 17640 \\ 0 & 7020 & 0 & -14040 \\ 0 & 0 & 3780 & 11340 \end{bmatrix}$$

$\boxed{\text{Fix the 2 and 3 row change 1 row with 2nd row}}$

$$\sim \begin{bmatrix} 88452000 & 0 & 0 & 88452000 \\ 0 & 7020 & 0 & -14040 \\ 0 & 0 & 3780 & 11340 \end{bmatrix} \quad R_1 \leftrightarrow 7020R_1 + 2520R_2$$

(e) $$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \therefore x = 1,\ y = -2,\ z = 3$$

(2) Solve the system of equations by Gauss - elimination method.

$$5x_1 + x_2 + x_3 + x_4 = 4$$

$$x_1 + 7x_2 + x_3 + x_4 = 12$$

$$x_1 + x_2 + 6x_3 + x_4 = -5$$

$$x_1 + x_2 + x_3 + 4x_4 = -6$$

Soln:

The given system is equivalent to

$$\begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 7 & 1 & 1 \\ 1 & 1 & 6 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ -5 \\ -6 \end{bmatrix}$$

$$[A, B] = \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 1 & 7 & 1 & 1 & 12 \\ 1 & 1 & 6 & 1 & -5 \\ 1 & 1 & 1 & 4 & -6 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 34 & 4 & 4 & 56 \\ 0 & 4 & 29 & 4 & -29 \\ 0 & 4 & 4 & 19 & -34 \end{array}\right] \quad \begin{array}{l} R_2 \leftrightarrow 5R_2 - R_1 \\ R_3 \leftrightarrow 5R_3 - R_1 \\ R_2 \leftrightarrow 5R_4 - R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 34 & 4 & 4 & 56 \\ 0 & 0 & 970 & 120 & -1210 \\ 0 & 0 & 120 & 630 & -1380 \end{array}\right] \quad \begin{array}{l} R_3 \rightarrow 34R_3 - 4R_2 \\ R_4 \rightarrow 34R_4 - 4R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 17 & 2 & 2 & 28 \\ 0 & 0 & 97 & 12 & -121 \\ 0 & 0 & 12 & 63 & -138 \end{array}\right] \quad \begin{array}{l} R_2 \rightarrow \frac{R_2}{2} \\ R_3 \rightarrow \frac{R_3}{10} \\ R_4 \rightarrow \frac{R_4}{10} \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 17 & 2 & 2 & 28 \\ 0 & 0 & 97 & 12 & -121 \\ 0 & 0 & 0 & 5967 & -11934 \end{array}\right] \quad R_4 \rightarrow 97R_4 - 12R_3$$

$$5967\, x_4 = -11934$$

$$x_4 = -2$$

$97x_3 + 12x_4 = -121$

$97x_3 + 12(-2) = -121$

$97x_3 - 24 = -121$

$97x_3 = -121 + 24$

$97x_3 = -97$

$\boxed{x_3 = -1}$

$17x_2 + 2x_3 + 2x_4 = 28$

$17x_2 + 2(-1) + 2(-2) = 28$

$17x_2 - 2 - 4 = 28$

$17x_2 - 6 = 28$

$17x_2 = 28 + 6$

$17x_2 = 34$

$\boxed{x_2 = 2}$

$5x_1 + x_2 + x_3 + x_4 = 4$

$5x_1 + 2 + (-1) + (-2) = 4$

$5x_1 - 1 = 4$

$5x_1 = 5$

$\boxed{x_1 = 1}$

Hence the solution is $x_1 = 1$, $x_2 = 2$, $x_3 = -1$, $x_4 = -2$

(3) using the Gauss-Jordan method solve the following equations

$10x + y + z = 12$

$2x + 10y + z = 13$

$x + y + 5z = 7$

Soln:

Interchanging the first and the last equation then

$$[A, B] = \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{array}\right]$$

Fix the pivot element row and make the other elements zero in the pivot element column.

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{array}\right] \quad \begin{array}{l} R_2 \leftrightarrow R_2 - 2R_1 \\ R_3 \leftrightarrow R_3 - 10R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & -9 & -49 & -58 \end{array}\right] \quad R_2 \leftrightarrow \frac{R_2}{8}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 6.125 & 7.125 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & -59.125 & -59.125 \end{array}\right] \quad \begin{array}{l} R_1 \leftrightarrow R_1 - R_2 \\ R_3 \leftrightarrow R_3 + 9R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 6.125 & 7.125 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_3 \leftrightarrow \frac{R_3}{-59.125}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \leftrightarrow R_1 - 6.125R_3 \\ R_1 \leftrightarrow R_2 + 1.125R_3 \end{array}$$

$\therefore x_1 = 1, \ y = 1, \ z = 1$

ITERATIVE METHODS

(a) Gauss Jacobi method

(b) Gauss-Seidal method.

① Solve the following system of equations by Gauss-Jacobi method and Gauss-Seidal method.

$$27x + 6y - z = 85$$
$$x + y + 54z = 110$$
$$6x + 15y + 2z = 72$$

Soln: As the coefficient matrix is not diagonally domiant we rewrite the equations.

$$27x + 6y - z = 85$$
$$x + y + 54z = 110$$
$$6x + 15y + 2z = 72$$

diagonally dominant

$\left[\because |a_{11}| > |a_{12}| + |a_{13}| ; \ |c_{33}| > |c_{31}| + |c_{32}| ; \ |b_{22}| > |b_{21}| + |b_{23}|\right]$

Since the diagonal elements are dominant in the coefficient matrix we write x, y, z as follows:

$$x = \frac{1}{27}\left[85 - 6y + z\right]$$

$$y = \frac{1}{15}\left[72 - 6x - 2z\right]$$

$$z = \frac{1}{54}\left[110 - x - y\right]$$

(i) Gauss Jacobi method

Let the initial values be $x=0$, $y=0$, $z=0$

First iteration:-

$$x^{(1)} = \frac{1}{27}[85] = 3.148$$
$$y^{(1)} = \frac{1}{15}[72] = 4.8$$
$$z^{(1)} = \frac{1}{54}[110] = 2.037$$

Second iteration:-

$$x^{(2)} = \frac{1}{27}[85 - 6y^{(1)} + z^{(1)}] = \frac{1}{27}[85 - 6.(4.8) + (2.037)] = 2.157$$
$$y^{(2)} = \frac{1}{15}[72 - 6x^{(1)} - 2z^{(1)}] = \frac{1}{15}[72 - 6(3.148) - 2(2.037)] = 3.269$$
$$z^{(2)} = \frac{1}{54}[110 - x^{(1)} - y^{(1)}] = \frac{1}{54}[110 - 3.148 - 4.8] = 1.890$$

Third iteration:-

$$x^{(3)} = \frac{1}{27}[85 - 6y^{(2)} + z^{(2)}] = \frac{1}{27}[85 - 6(3.269) + 1.890] = 2.492$$
$$y^{(3)} = \frac{1}{15}[72 - 6x^{(2)} + 2z^{(2)}] = \frac{1}{15}[72 - 6(2.157) - 2(1.890)] = 3.685$$
$$z^{(3)} = \frac{1}{54}[110 - x^{(2)} - y^{(2)}] = \frac{1}{54}[110 - 2.157 - 3.269] = 1.937$$

Fourth iteration:-

$$x^{(4)} = \frac{1}{27}[85 - 6y^{(3)} + z^{(3)}] = \frac{1}{27}[85 - 6(3.685) + 1.937] = 2.401$$
$$y^{(4)} = \frac{1}{15}[72 - 6x^{(3)} - 2z^{(3)}] = \frac{1}{15}[72 - 6(2.492) - 2(1.937)] = 3.545$$
$$z^{(4)} = \frac{1}{54}[110 - x^{(3)} - y^{(3)}] = \frac{1}{54}[110 - 2.492 - 3.685] = 1.923$$

Fifth iteration:-

$$x^{(5)} = \frac{1}{27}[85 - 6y^{(4)} + z^{(4)}] = \frac{1}{27}[85 - 6(3.545) + 1.923] = 2.432$$
$$y^{(5)} = \frac{1}{15}[72 - 6x^{(4)} - 2z^{(4)}] = \frac{1}{15}[72 - 6(2.401) - 2(1.923)] = 3.583$$
$$z^{(5)} = \frac{1}{54}[110 - x^{(4)} - y^{(4)}] = \frac{1}{54}[110 - 2.401 - 3.645] = 1.927$$

Sixth Iteration

$$x^{(6)} = \frac{1}{27}[85 - 6y^{(5)} + z^{(5)}] = \frac{1}{27}[85 - 6(3.583) + 1.927] = 2.423$$
$$y^{(6)} = \frac{1}{15}[72 - 6x^{(5)} - 2z^{(5)}] = \frac{1}{15}[72 - 6(2.4332) - 2(1.927)] = 3.570$$

$$z^{(6)} = \frac{1}{54}\left[110 - x^{(5)} - y^{(5)}\right] = \frac{1}{54}\left[110 - 2.432 - 3.583\right] = 1.926$$

Seventh iteration:

$$x^{(7)} = \frac{1}{27}\left[85 - 6y^{(6)} + z^{(6)}\right] = \frac{1}{27}\left[85 - 6(3.570) + 1.926\right] = 2.426$$

$$y^{(7)} = \frac{1}{15}\left[72 - 6x^{(6)} - 2z^{(6)}\right] = \frac{1}{15}\left[72 - 6(2.423) - 2(1.926)\right] = 3.574$$

$$z^{(7)} = \frac{1}{54}\left[110 - x^{(6)} - y^{(6)}\right] = \frac{1}{54}\left[110 - 2.423 - 3.570\right] = 1.926$$

Eighth Iteration:

$$x^{(8)} = \frac{1}{27}\left[85 - 6y^{(7)} + z^{(7)}\right] = \frac{1}{27}\left[85 - 6(3.574) + 1.926\right] = 2.425$$

$$y^{(8)} = \frac{1}{15}\left[72 - 6x^{(7)} - 2z^{(7)}\right] = \frac{1}{15}\left[72 - 6(2.426) - 2(1.926)\right] = 3.573$$

$$z^{(8)} = \frac{1}{54}\left[110 - x^{(7)} - y^{(7)}\right] = \frac{1}{54}\left[110 - 2.426 - 3.574\right] = 1.926$$

Nineth Iteration:

$$x^{(9)} = \frac{1}{27}\left[85 - 6y^{(8)} + z^{(8)}\right] = \frac{1}{27}\left[85 - 6(3.573) + 1.926\right] = 2.426$$

$$y^{(9)} = \frac{1}{15}\left[72 - 6x^{(8)} - 2z^{(8)}\right] = \frac{1}{15}\left[72 - 6(2.425) - 2(1.926)\right] = 3.573$$

$$z^{(9)} = \frac{1}{54}\left[110 - x^{(8)} - y^{(8)}\right] = \frac{1}{54}\left[110 - 2.425 - 3.573\right] = 1.926$$

Tenth Iteration:

$$x^{(10)} = \frac{1}{27}\left[85 - 6y^{(9)} + z^{(9)}\right] = \frac{1}{27}\left[85 - 6(3.573) + 1.926\right] = 2.426$$

$$y^{(10)} = \frac{1}{15}\left[72 - 6x^{(9)} - 2z^{(9)}\right] = \frac{1}{15}\left[72 - 6(2.426) - 2(1.926)\right] = 3.573$$

$$z^{(10)} = \frac{1}{54}\left[110 - x^{(9)} - y^{(9)}\right] = \frac{1}{54}\left[110 - 2.426 - 3.573\right] = 1.926$$

Hence $x = 2.426$, $y = 3.573$, $z = 1.926$

[Correct to three decimal places]

(2) Gauss - seidel method.

Let the initial values be $y = 0$ $z = 0$.

First iteration

$x^{(1)} = \frac{1}{27}[85 - 6y^{(0)} + z^{(0)}] = \frac{1}{27}[85 - 6(0) + 0] = 3.148$

$y^{(1)} = \frac{1}{15}[72 - 6x^{(1)} - 2z^{(0)}] = \frac{1}{15}[72 - 6(3.148) - 0] = 3.541$

$z^{(1)} = \frac{1}{54}[110 - x^{(1)} - y^{(1)}] = \frac{1}{54}[110 - 3.148 - 3.541] = 1.913$

second iteration:-

$x^{(2)} = \frac{1}{27}[85 - 6y^{(1)} + z^{(1)}] = \frac{1}{27}[85 - 6(3.541) + 1.913] = 2.432$

$y^{(2)} = \frac{1}{15}[72 - 6x^{(2)} - 2z^{(1)}] = \frac{1}{15}[72 - 6(2.432) - 2(1.913)] = 3.572$

$z^{(2)} = \frac{1}{54}[110 - x^{(2)} - y^{(2)}] = \frac{1}{54}[110 - 2.432 - 3.572] = 1.926$

Third iteration:

$x^{(3)} = \frac{1}{27}[85 - 6y^{(2)} + z^{(2)}] = \frac{1}{27}[85 - 6(3.572) + 1.926] = 2.426$

$y^{(3)} = \frac{1}{15}[72 - 6x^{(3)} - 2z^{(2)}] = \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573$

$z^{(3)} = \frac{1}{54}[110 - x^{(3)} - y^{(3)}] = \frac{1}{54}[110 - 2.426 - 3.573] = 1.926$

Fourth iteration:-

$x^{(4)} = \frac{1}{27}[85 - 6y^{(3)} + z^{(3)}] = \frac{1}{27}[85 - 6(3.573) + 1.926] = 2.426$

$y^{(4)} = \frac{1}{15}[72 - 6x^{(4)} - 2z^{(3)}] = \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573$

$z^{(4)} = \frac{1}{54}[110 - x^{(4)} - y^{(4)}] = \frac{1}{54}[110 - 2.426 - 3.573] = 1.926$

Hence $x = 2.426$, $y = 3.573$, $z = 1.926$

This shows that the convergence is rapid in Gauss-Seidel method when compared to Gauss-Jacobi method.

(3) Solve the following equations by Gauss-Seidel method.

$4x + 2y + z = 14$

$x + 5y - z = 10$

$x + y + 8z = 20$

Soln:-

as the coefficient matrix is diagonally dominant solving for x, y, z we get

$$x = \frac{1}{4}[14 - 2y - z]$$

$$y = \frac{1}{5}[10 - x + z]$$

$$z = \frac{1}{8}[20 - x - y]$$

let the initial values be $y=0$, $z=0$

First iteration:-

$$x^{(1)} = \frac{1}{4}[14 - 2(0) - (0)] = \frac{14}{4} = 3.5$$

$$y^{(1)} = \frac{1}{5}[10 - x^{(1)} + z^{(0)}] = \frac{1}{5}[10 - 3.5 + 0] = 1.3$$

$$z^{(1)} = \frac{1}{8}[20 - x^{(1)} - y^{(1)}] = \frac{1}{8}[20 - 3.5 - 1.3] = 1.9$$

Second iteration:-

$$x^{(2)} = \frac{1}{4}[14 - 2y^{(1)} - z^{(1)}] = \frac{1}{4}[14 - 2(1.3) - (1.9)] = 2.375$$

$$y^{(2)} = \frac{1}{5}[10 - x^{(2)} + z^{(1)}] = \frac{1}{5}[10 - 2.375 + 1.9] = 1.905$$

$$z^{(2)} = \frac{1}{8}[20 - x^{(2)} - y^{(2)}] = \frac{1}{8}[20 - 2.375 - 1.905] = 1.965$$

Third iteration:-

$$x^{(3)} = \frac{1}{4}[14 - 2y^{(2)} - z^{(2)}] = \frac{1}{4}[14 - 2(1.905) - 1.965] = 2.056$$

$$y^{(3)} = \frac{1}{5}[10 - x^{(3)} + z^{(2)}] = \frac{1}{5}[10 - 2.056 + 1.965] = 1.9818$$

$$z^{(3)} = \frac{1}{8}[20 - x^{(3)} - y^{(3)}] = \frac{1}{8}[20 - 2.056 + 1.9818] = 1.995$$

Fourth iteration:-

$$x^{(4)} = \frac{1}{4}[14 - 2y^{(3)} - z^{(3)}] = \frac{1}{4}[14 - 2(1.9818) - 1.965] = 2.510$$

$$y^{(4)} = \frac{1}{5}[10 - x^{(4)} + z^{(3)}] = \frac{1}{5}[10 - 2.510 + 1.995] = 1.897$$

$$z^{(4)} = \frac{1}{8}[20 - x^{(4)} - y^{(4)}] = \frac{1}{8}[20 - 2.510 - 1.897] = 1.949$$

Fifth iteration:-

$$x^{(5)} = \frac{1}{4}[14 - 2y^{(4)} - z^{(4)}] = \frac{1}{4}[14 - 2(1.897) - 1.949] = 2.064$$

$$y^{(5)} = \frac{1}{5}[10 - x^{(5)} + z^{(4)}] = \frac{1}{5}[10 - 2.064 + 1.949] = 1.977$$

$$z^{(5)} = \frac{1}{8}[20 - x^{(5)} - y^{(5)}] = \frac{1}{8}[20 - 2.064 - 1.977] = 1.995$$

Sixth iteration:-

$$x^{(6)} = \frac{1}{4}[14 - 2y^{(5)} - z^{(5)}] = \frac{1}{4}[14 - 2(1.977) - 1.995] = 2.013$$

$$y^{(6)} = \frac{1}{5}[10 - x^{(6)} + z^{(5)}] = \frac{1}{5}[10 - 2.013 + 1.995] = 1.996$$

$$z^{(6)} = \frac{1}{8}[20 - x^{(6)} - y^{(6)}] = \frac{1}{8}[20 - 2.013 - 1.996] = 1.999$$

Seventh iteration:-

$$x^{(7)} = \frac{1}{4}[14 - 2y^{(6)} - z^{(6)}] = \frac{1}{4}[14 - 2(1.996) - 1.999] = 2.002$$

$$y^{(7)} = \frac{1}{5}[10 - x^{(7)} + z^{(6)}] = \frac{1}{5}[10 - 2.002 + 1.999] = 1.999$$

$$z^{(7)} = \frac{1}{8}[20 - x^{(7)} - y^{(7)}] = \frac{1}{8}[20 - 2.002 - 1.999] = 2.000$$

Eighth iteration:-

$$x^{(8)} = \frac{1}{4}[14 - 2y^{(7)} - z^{(7)}] = \frac{1}{4}[14 - 2(1.999) - 2] = 2.001$$

$$y^{(8)} = \frac{1}{5}[10 - x^{(8)} + z^{(7)}] = \frac{1}{5}[10 - 2.001 + 2] = 2.000$$

$$z^{(8)} = \frac{1}{8}[20 - x^{(8)} - y^{(8)}] = \frac{1}{8}[20 - 2.001 - 2] = 2.000$$

Ninth iteration:-

$$x^{(9)} = \frac{1}{4}[14 - 2y^{(8)} - z^{(8)}] = \frac{1}{4}[14 - 2(2) - 2] = 2$$

$$y^{(9)} = \frac{1}{5}[10 - x^{(9)} + z^{(8)}] = \frac{1}{5}[10 - 2 + 2] = 2$$

$$z^{(9)} = \frac{1}{8}[20 - x^{(9)} - y^{(9)}] = \frac{1}{8}[20 - 2 - 2] = 2$$

Tenth iteration:-

$$x^{(10)} = \frac{1}{4}[14 - 2y^{(9)} - z^{(9)}] = \frac{1}{4}[14 - 2(2) - 2] = 2$$

$$y^{(10)} = \frac{1}{5}[10 - x^{(10)} + z^{(9)}] = \frac{1}{5}[10 - 2 + 2] = 2$$

$$z^{(10)} = \frac{1}{8}[20 - x^{(10)} - y^{(10)}] = \frac{1}{8}[20 - 2 - 2] = 2$$

Hence $x = 2$, $y = 2$, $z = 2$

INVERSE OF A MATRIX BY GAUSS JORDON METHOD

Gauss - Jordan elimination method.

① using Gauss-Jordan method, find the inverse of the matrix.

$$\begin{bmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$$

Soln:-

$$[A, I] = \left[\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3/2 & 1/2 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 5 & 0 & 1 & 1 \end{array}\right] \quad R_1 \rightarrow \frac{R_1}{2}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3/2 & 1/2 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & 2 & 7/2 & -1/2 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \leftrightarrow R_2 - 2R_1 \\ R_3 \leftrightarrow R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 2 & 7/2 & -1/2 & 0 & 1 \end{array}\right] \quad R_2 \leftrightarrow \frac{R_2}{-1}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & -1/2 & -5/2 & 2 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \leftrightarrow R_1 - R_2 \\ R_3 \leftrightarrow R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1/2 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 5 & -4 & -2 \end{array}\right] \quad R_3 \leftrightarrow R_3(-2)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & -9 & 7 & 4 \\ 0 & 0 & 1 & 5 & -4 & -2 \end{array}\right] \quad \begin{array}{l} R_1 \leftrightarrow R_1 + \frac{1}{2}R_3 \\ R_2 \leftrightarrow R_2 - 2R_3 \end{array}$$

Hence $A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ -9 & 7 & 4 \\ 5 & -4 & -2 \end{bmatrix}$

verification:-

$$AA^{-1} = I$$

$$\begin{bmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ -9 & 7 & 4 \\ 5 & -4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

② using Gauss - Jordan method, find the inverse of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Soln:

Let $[A, I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$ $R_2 \leftrightarrow R_2 - R_1$, $R_3 \leftrightarrow R_3 + 2R_1$

$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$ $R_2 \leftrightarrow \frac{R_2}{2}$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 3/2 & -1/2 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right]$ $R_1 \leftrightarrow R_1 - R_2$, $R_3 \leftrightarrow R_3 + 2R_2$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 3/2 & -1/2 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$ $R_3 \leftrightarrow \frac{R_3}{-4}$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$ $R_1 \leftrightarrow R_1 - 6R_3$, $R_2 \leftrightarrow R_2 + 3R_3$

$$A^{-1} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$$

verification:-

$$AA^{-1} = I$$

$$= \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{Ans.}$$

EIGEN VALUE OF A MATRIX BY POWER METHOD

The power method

① Find the numerically largest eigenvalue of $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ by power method.

Soln:-

Let $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an arbitrary initial eigenvector.

$$AX_1 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 0.167 \\ 0.667 \\ 1 \end{bmatrix} = 6X_2$$

$$AX_2 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.167 \\ 0.667 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.166 \\ 2.336 \\ 8.003 \end{bmatrix} = 8.003 \begin{bmatrix} 0.021 \\ 0.292 \\ 1 \end{bmatrix} = 8.003\, X_3$$

$$0.167 - 3(0.067) + 2(1) = 0.166$$

$$4(0.167) + 4(0.667) - 1 = 2.336$$

$$6(0.167) + 3(0.667) + 5 = 8.003$$

$$AX_3 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.021 \\ 0.292 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.145 \\ 0.252 \\ 6.002 \end{bmatrix} = 6.002 \begin{bmatrix} 0.191 \\ 0.042 \\ 1 \end{bmatrix} = 6.002\, X_4$$

$$1(0.021) - 3(0.292) + 2(1) = 1.145$$

$$4(0.021) + 4(0.292) - 1(1) = 0.252$$

$$6(0.021) + 3(0.292) + 5(1) = 6.002$$

$$AX_4 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.191 \\ 0.042 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.065 \\ -0.068 \\ 6.272 \end{bmatrix} = 6.272 \begin{bmatrix} 0.329 \\ -0.011 \\ 1 \end{bmatrix} = 6.272\, X_5$$

$$1(0.191) - 3(0.042) + 2(1) = 2.065$$

$$4(0.191) + 4(0.042) - 1(1) = -0.068$$

$$6(0.191) + 3(0.042) + 5(1) = 6.272$$

$$AX_5 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.329 \\ -0.011 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.362 \\ 0.272 \\ 6.941 \end{bmatrix} = 6.941 \begin{bmatrix} 0.34 \\ 0.039 \\ 1 \end{bmatrix} = 6.941\, X_6$$

$$AX_6 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.34 \\ 0.039 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.223 \\ 0.516 \\ 7.157 \end{bmatrix} = 7.157 \begin{bmatrix} 0.311 \\ 0.072 \\ 1 \end{bmatrix} = 7.157\, X_7$$

$$AX_7 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.311 \\ 0.072 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.095 \\ 0.532 \\ 7.082 \end{bmatrix} = 7.082 \begin{bmatrix} 0.296 \\ 0.075 \\ 1 \end{bmatrix} = 7.082\, X_8$$

$$AX_8 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.296 \\ 0.075 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.071 \\ 0.484 \\ 7.001 \end{bmatrix} = 7.001 \begin{bmatrix} 0.296 \\ 0.069 \\ 1 \end{bmatrix} = 7.001\, X_9$$

This shows the largest eigenvalue $= 7$.

② Find the dominant eigen value and the corresponding eigen vector of $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ find also the least latent root and hence the third value also.

Soln:

Let $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be an approximate eigen value.

$$AX_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\, X_2$$

$$AX_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = 7 \cdot X_3$$

$$AX_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.5714 \\ 1.8572 \\ 0 \end{bmatrix} = 3.5714 \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = 3.5714\, X_4$$